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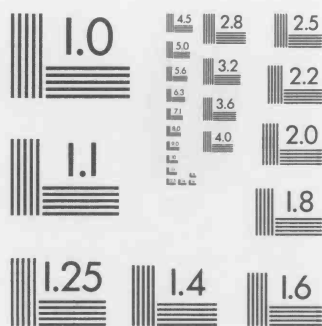
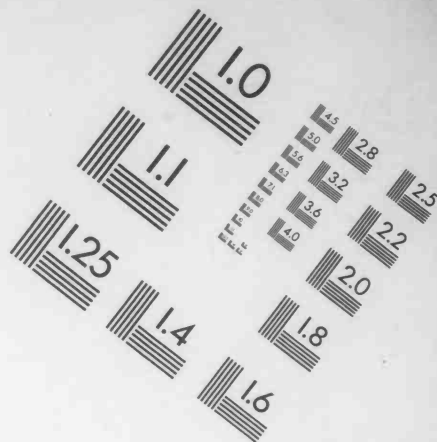
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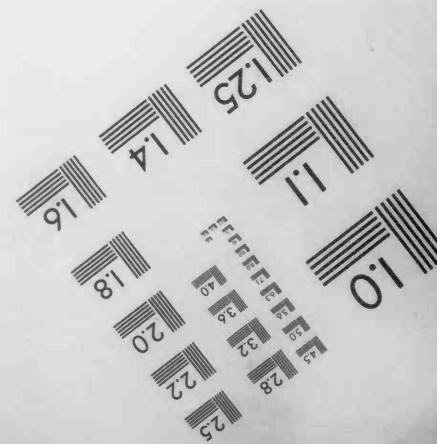
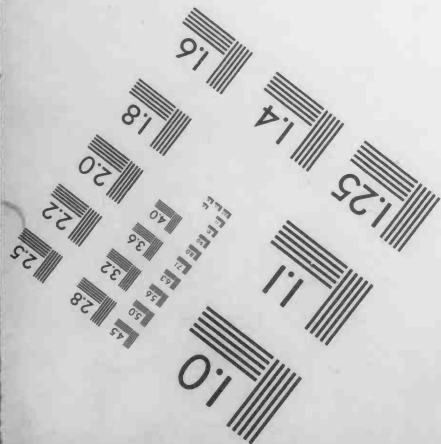
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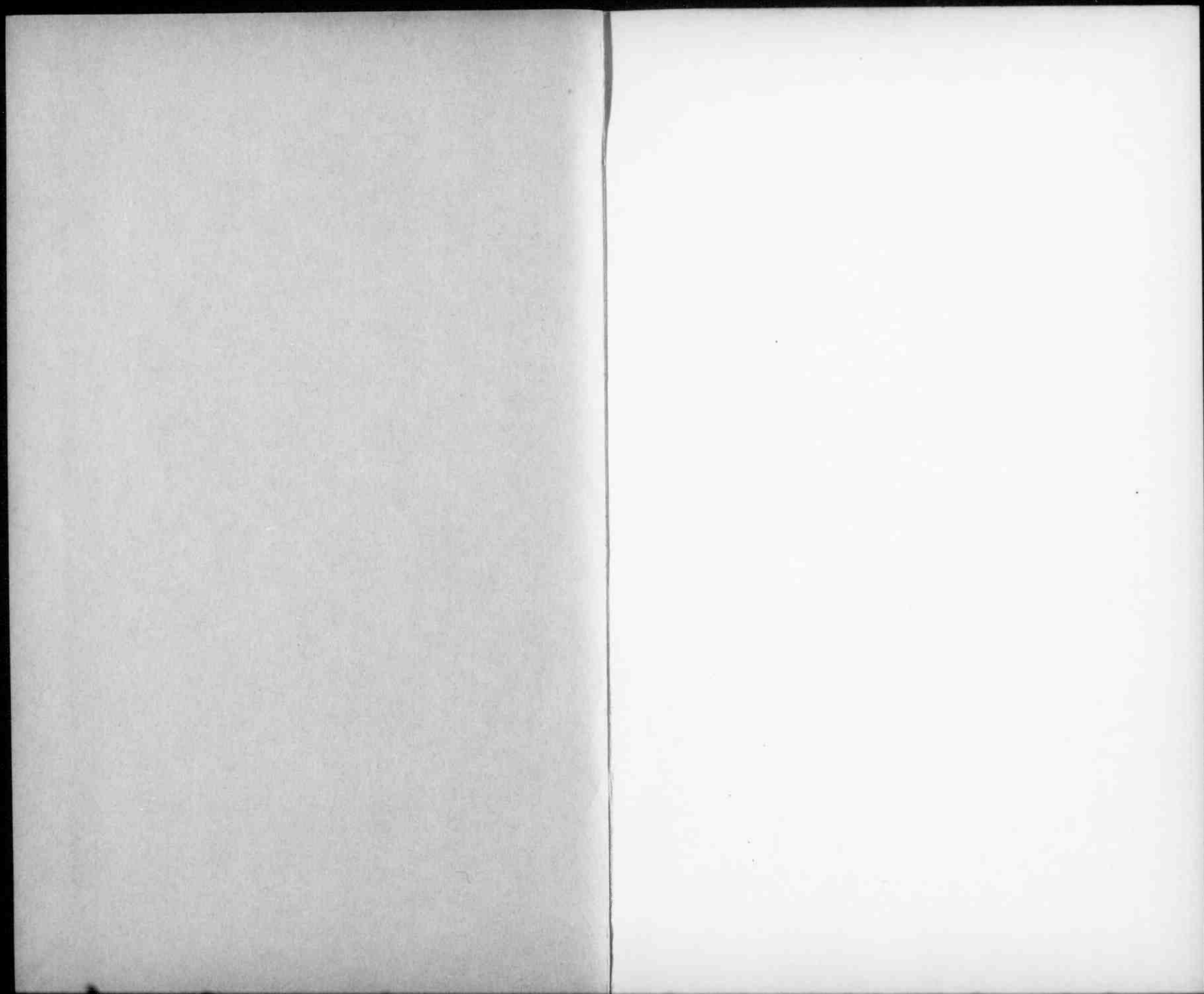
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# LOGIC

## PART I



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# LOGIC

## PART I

BY

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ΠΑΝΤΑ ΡΕΙ ΕΙ ΜΗ ΤΟ ΑΛΗΘΕΣ

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'MAN' <sup>is</sup> ~~is~~ A "RATIONAL ANIMAL"

Definition

Corrigenda

p. 24  
p. 32  
p. 145

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## PREFACE

THE present work is intended to cover the whole field of Logic as ordinarily understood. It includes an outline of elementary Formal Logic, which should be read in close connection with Dr Keynes's classical work, in which the last word has been said on most of the fundamental problems of the subject. As regards Material Logic, I have taken Mill's *System of Logic* as the first basis of discussion, which however is subjected to important criticisms mostly on the lines of the so-called conceptualist logicians.

I have to express my great obligations to my former pupil, Miss Naomi Bentwich, without whose encouragement and valuable assistance in the composition and arrangement of the work, it would not have been produced in its present form.

W. E. J.

March 30, 1921.

Part I. - Formal Logic

Part II. - Demonstration

Part III. - Methodology

Part IV. - Probability

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## INTRODUCTION

§ 1. LOGIC is most comprehensively and least controversially defined as the analysis and criticism of thought. This definition involves the least possible departure from the common understanding of the term and is not intended to restrict or extend its scope in any unusual way. The scope of logic has tended to expand in two directions—backwards into the domain of metaphysics, and forwards into that of science. These tendencies show that no rigid distinction need be drawn on the one side between logic and metaphysics, nor on the other between logic and science. The limits imposed by any writer are justified so far as his exposition exhibits unity; it is, in fact, much more important to remove confusions and errors within the subjects discussed under the head of logic, than to assign precise limits to its scope. It is, I hold, of less importance to determine the line of demarcation between logic and philosophy than that between logic and science; so that my treatment of logic might be called philosophical in comparison with that of those who implicitly or explicitly separate their criticism and analysis from what in their view should be relegated to epistemology and ontology.

This account of the scope of logic does not differ in any essential respects from that given, for example, in Mill's long introductory chapter. The special feature of Mill's logic is the great prominence given to the theory of induction, in contrast to most of his predecessors

and contemporaries, including Whately. Whately does not omit reference to induction any more than Mill omits syllogism: where they differ is that Whately asserts that in order to be valid any inductive inference must be formulated syllogistically, and that therefore the principle for induction is dependent on the principle of syllogism. Mill opposes this view; but as regards the scope of logic there is no disagreement between them: they differ simply on the question of the relations of deduction to induction.

If any writer deliberately or on principle dismisses from logic the theory of inductive inference, it must be on one of three grounds: either (*a*) that no inductive inference is valid; or (*b*) that different criteria of validity apply to different sciences; or (*c*) that the problem of the validity of induction constitutes a topic to be included in some study other than that named logic. As regards (*a*), this is the view which seems to be held by Venn in his *Empirical Logic* where, in the chapter on the subjective foundations of induction, he acknowledges that as a matter of fact human beings do make directly inductive inferences, even with a feeling of conviction, but that no warrant for such conviction can be found. Another aspect of his view of induction is expounded in the chapter on the objective foundations of induction, in which he classifies the different kinds of uniformity—such as sequence, co-existence, permanence, rhythm—which are used as major premisses, expressive of actual fact, by means of which specific uniformities under each general head are established as valid. When then he is asked what reasonable ground there is for accepting these major premisses as true, he

maintains in effect that they have to be assumed, in order to give security to the conclusions inductively inferred. In using the word assumption, there seems to be some ambiguity, namely whether it is to be understood to mean 'assumed to be true although known to be false' or 'assumed to be true although unprovable.' I take Venn to mean the latter, and that the attitude towards this assumption is merely one of felt certainty—felt, indeed, by all human beings, but having no root in our rational nature, and only exhibiting a common psychological disposition or character. This view, that there is no inductive principle that is self-evidently or demonstrably true, seems to be held by many other logicians, though none of them, I think, put it as explicitly as Venn. So while he and others include induction in their logical exposition, they neglect what I take to be the essential justification for its inclusion, namely as affording a systematic criticism of the question of its validity. As regards (*b*), many excellent text-books have been written in these days treating of the principles and methods peculiar to different sciences; it is not denied by their authors that this treatment is logical; but, if not explicitly stated, yet it seems to be suggested that in comparing the logic of one science with that of another the sole result is to exhibit differences, and that no one set of principles applies to all the different sciences. If this were the fact there would be some excuse for excluding the treatment of induction from the scope of logic, on the ground that the discussion of each of the separate principles should be relegated to its own department of science. But if, as I hold in agreement with most

*of Hume*

other logicians, there must be a community of principle discoverable in all sciences, then the discussion of this must be included in logic. As regards (c) the question raised seems to be: 'Given the topic induction, what name shall be given to the science that includes it in its treatment?' rather than the converse question 'Given the name logic, shall it be defined so as to include, or so as to exclude, induction?' If we put the question in the first form, the answer is of course purely arbitrary; we might give it the name Epagogics. But if the question is put in the second form, the answer is not in the same sense arbitrary, assuming that there is general unanimity as regards the usage of the name logic to denote a science whose central or essential function is to criticise thought as valid or invalid. That induction should be included in logic thus defined follows from the undeniable fact that we do infer inductively, and that some persons in reference to some problems do infer invalidly. Even if this were not the fact, it is certainly of scientific importance to render explicit what everyone implicitly recognises in their inferences—as much for the case of induction as for that of syllogism or other formal types of inference. It has even been hinted that nobody makes mistakes in formal inference; and yet—in despite of this, if true—no one questions the value of systematising the principles under which people may unconsciously reason; and what holds of formal inference would certainly hold *à fortiori* of the processes of inductive inference which present many more serious opportunities for fallacy.

§ 2. As regards the term 'thought' which enters

into my definition, its application is intended to include perceptual judgments which are commonly contrasted with rather than subsumed under thought, for the reason that thought is conceived as purely abstract while perception contains an element of concreteness. But properly speaking even in perceptual judgment there is an element of abstraction; and on the other hand no thought involves mere abstraction. It follows, therefore, that the processes of thinking and of perceptual judgment have an essential identity of character which justifies their treatment in a single systematic whole. It is the distinction between sense-experience and perceptual judgment, and not that between perceptual judgment and thought, that must be emphasised. The essential feature of perceptual judgment in contrast to mere sense-experience is that it involves activity, and that this activity is controlled by the purpose of attaining truth; further it is the presence of this purpose which distinguishes thought from other forms of mental activity. Thought may therefore be defined as mental activity controlled by a single purpose, the attainment of truth.

§ 3. Now it is true, as often urged, that thought is motivated not solely by the purpose of attaining truth, but rather by the intention of realising a particular end in some specific form and under certain specific circumstances. But I have to maintain that any other or further purpose which may prompt us to undertake the activity of thinking is irrelevant to the nature of thought as such, this other purpose serving only to determine the direction of activity. When such activity is actually in operation its course is wholly independent of the



prompting motive and guided by the single purpose of attaining truth. For instance, our desire for food may prompt us to search for it; but this resolve, once taken, leads to a thinking process the purpose of which is to come to some conclusion as to where food is likely to be found, and the sole aim of this process is to discover what is true on the matter in hand. This being so, the logical treatment of thought must be disencumbered from all reference to any ulterior purpose.

Whether truth is ever pursued without any ulterior purpose is a psychological question which may fairly be asked; and if introspection is to be trusted must certainly be answered in the affirmative, although the enquiry whether true knowledge has intrinsic value or not belongs to ethics. That the attainment of truth for its own sake constitutes a genuine motive force is further confirmed by recognising the fact that people do actually attach value to true knowledge, as is incontestably proved by their willingness to defy the prospect of social disapprobation, persecution, and even martyrdom incurred by the utterance and promulgation of what they hold to be true. At the same time, it must be pointed out that the aim of the thinking process is not the attainment of truth in general, but always of truth in regard to some determinate question under consideration. This is closely analogous to the psychological fact that what we desire is never pleasure in general, but always—if the doctrine of psychological hedonism is to be accepted—some specific experience which is represented as pleasurable.

Any thinking process is normally initiated by a question and terminated by an answer; what dis-

tinguishes one thinking process from another is the difference of the question proposed. The bond of unity amongst the phases of a single process does not necessarily entail unbroken temporal continuity, but only identity of the question proposed. Indeed any thought process may be temporarily interrupted before the proposed question has been answered. It must be left as a topic for psychology to investigate the causes of such suspension, and how far the advance made serves as a starting point for further advances. Logic, on the other hand, is concerned with the nature of the advance as an advance and criticises the process from the point of view of validity or invalidity.

§ 4. The above definition of logic as the analysis and criticism of thought should be compared with that of the Scholastics, who laid emphasis on the point that logic is concerned with the art of thinking, where art is nearly equivalent to the modern term technique, and has an understood reference to activity with an end in view. The study of the art of thinking as thus understood is of use in instructing us how to proceed when thinking out any problem: for instance, it lays down rules of classification and division for the clearing up of obscurities and inconsistencies in thought; rules for the recall and selection of knowledge appropriate to any given problem; etc. Descartes' *Discourse on Method* is a classical illustration of this species of science. Modern examples of excellent treatises on these lines are to be found in Alfred Sidgwick, and other neo-pragmatists. It is a science of the highest value, and need only be separated from logic on the ground of the difference of purpose; inasmuch as its direct

purpose is the attainment of valid thought, whereas logic is the study of the conditions of valid thought, and as such it does not exclude the study of the art.

§ 5. Alongside of the use of the term 'art' to mean technique, there is a more modern usage where it implies reference to aesthetic feelings and judgments. Nowadays discussions as to whether an objective standard for these feelings and judgments should be recognised are very prominent. The nature of the feelings and judgments that enter into aesthetic appreciation belongs to psychology; but if we agree that there is a discoverable objective standard, then the treatment of the subject of aesthetics is to be distinguished from the psychological treatment, precisely as the treatment of thought in logic is distinguished from that in psychology.

Aesthetics, in this sense, raises very similar problems to those presented in Ethics; and it is frequently said that as normative Logic, Aesthetics and Ethics are related in the same way to the three psychological factors, thought, feeling and volition respectively. Each of the normative studies may be said to be based on a standard of value, the precise determination of which it is their function to formulate; in each, imperatives are laid down which are acknowledged by the individual, not on any external authority, but as self-imposed; and, in each, the ultimate appeal is to the individual's intuitive judgment. There is, however, a closer resemblance between Ethics and Aesthetics in their relations to volitions and feelings respectively, than between either of them and Logic; inasmuch as there are apparently fundamental differences of opinion

as to the ultimate ethical and aesthetical standards, that give to the studies of Ethics and Aesthetics a controversial character absent from Logic about whose standards there is no genuine disagreement. As regards the relation of Ethics to Logic, the question sometimes arises as to which subject is supreme. The answer to this question depends entirely upon the nature of the supremacy intended: the imperatives for thought become imperatives for conduct only on condition that true judgments have intrinsic value and false judgments intrinsic disvalue; and thus, from the point of view of conduct, Logic is subordinate to Ethics. On the other hand, ethical enquiry—like any other scientific investigation—has to avoid violating logical principles, so that from the point of view of true thought Logic is supreme over Ethics.

§ 6. Our discussion so far has led us to consider the relations of Logic to Philosophy in general, Psychology, Aesthetics and Ethics. Another subject to which it is closely allied and from which it is yet distinct is Grammar, the alliance being *prima facie* accounted for by the common concern of the two studies with language. The connection between thought and language presents a problem for the science of Psychology; but, so far as thinking or the communication of thought involves the use of words, the provinces of Logic and Grammar coincide; that is to say universal Grammar, which excludes what pertains to different languages and includes only what is common to all languages, should be subsumed under Logic. For the modes in which words are combined—which constitute the subject matter for Grammar—cannot be expounded or

understood except as reflecting the modes in which thoughts are combined; and this combination is effected by means of such logical operations as negation, conjunction, disjunction, alternation, implication and so on, represented by the words *not*, *and*, *not both*, *or*, *if*, etc. To justify the subordination of Grammar to Logic we have only to realise that the analysis of the sentence in Grammar corresponds to the analysis of thought in Logic, and that grammatical criticism is confined to securing that the sentence precisely represents the thought, any further criticism of the proposition coming exclusively within the province of Logic. It may be pointed out in this connection as specially significant both for the linguist and for the logician, that languages differ in the degree of their capacity to exhibit through their structure intimacy between words and thoughts.

§ 7. Amongst all the sciences over which logic must rule, there is one that occupies a unique place. The constituents of thought which are in the most narrow sense logical are those which give form to the construct, connecting alien elements by modes which give specific significance to the whole. The first group of these is expressed by ties, conjunctive words, prepositional words, and modes of verbal inflection. But as the form of thought is further elaborated there enter new kinds of terms, namely specific adjectives which have a constant meaning definable in terms of pure thought, or else are to be admitted and understood as indefinables. The most generic form of such adjectives directly expresses the result of such mental acts of comparison as like, unlike, different from, agreeing with. Owing to the purely logical nature of these relations, universal

formulae in which they are introduced can be constructed by mere abstract thought. The preliminary condition for this construction is the separating of what is given to constitute a plurality, and thus to introduce a formal factor which can only be verbally expressed by the separations and juxtapositions of the substantial words. The very general relation that separation effects is that most indeterminate relation *otherness*. When the complementary notions of separateness and togetherness are joined to constitute a unity, there enters the idea of number, and we are in the domain of mathematics.

The extraordinary capacity for development that marks mathematics is due to the precision with which the relations of comparison are capable of being amplified. Through the substitutions that are thus rendered possible, the range of application of mathematical formulae is extended beyond the bounds which would otherwise delimit logic. Any material that might be presented to thought upon which the same precise operations of comparison could be performed, would lead to the same forms as mathematics. For example ideas, not only of difference, but of determinable degrees of difference, bring the material into relations of intrinsic order, and out of these relations emanate relations between relations, so that theoretically the science develops into a highly complicated system. The point then, where we may venture to say that logic actually passes into mathematics is where the specific indefinable adjectives above referred to give new material for further logical combinations.

Here it is of great importance to point to the

relative nature of the distinction between form and matter. Logic begins with a sharp contrast between matter, as what is given as merely shapeless, and form, as that which thought imposes. But as we advance to mathematics, we impose a new element of form in introducing the relation otherness and its developments; and this being operated on by thought takes the place of new matter: in short, what is introduced as matter is form in the making. All this could be summed up by saying that for elemental logic, mathematical notions would constitute matter; whereas when the step into mathematics is once taken these same elements are just those in accordance with which thought advances in constructing more and more complicated forms. This view of the relation of logic to mathematics will be worked out in Part II of the present work under 'Demonstration,' where the procedure of building up mathematical science is shown to involve the very same principles as are used in the logical structure.

*Part II.*

All the sciences, including mathematics, over which logic has supreme control, have been properly described as applied logic. But mathematics is applied logic in a certain very unique sense, for mathematics is nothing but an extension of logical formulae introducing none but purely logical factors; while every other science borrows its material from experiential sources, and can only use logical principles when or after such material is supplied. Within mathematics we have again the same kind of distinction, namely that between pure and applied mathematics, as it has been called. In pure mathematics, the mathematician can give free

play to his imagination in constructing forms that are restricted only by principles of logical consistency, and he develops the implications that are derivable from what may be indifferently regarded either as definitions of his fictitious constructs or as hypothetically entertained first axioms. In order that these axioms and the theorems therefrom derived may be considered as true, recourse must be had to the real world, and if applicable, the axioms come to be assertorically entertained as premisses, and the derived propositions as the developed conclusions. This application of mathematics to reality constitutes applied mathematics. Taking geometry as our first example, while there is no limit to constructing conceived spaces other than Euclidian, their application to reality demands the enquiry whether our space is or is not Euclidian. This is answered by an appeal to our immediate intuitions directed to our spatial experiences, and it is this appeal that is outside the range of pure mathematics. Again the merely logical conception of betweenness, which develops into that of serial orders of lower or higher forms of complexity, is in the first instance a product of pure logical constructiveness, and would yield implications from which a system of implicates could be developed. But such a hypothetically conceived body of propositions would have no basis in the real but for the applicability of the defined conceptions to what is given in non-mathematical intuition. This applicability holds not only in the domain of spatial order, but also in that of the qualitative relations of difference which impose serial order amongst sense impressions.

Regarded in the light of its control over all sciences

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logic has been called by the name 'Methodology'; that is to say while the forms of logic implicitly control the conclusions of science, logic itself includes the study which renders explicit the ways according to which its authority is exercised. The department of logic known as methodology constitutes the third part of the present work, which is entitled 'The logical foundations of Science.'

Part III

Another illustration of applied mathematics is to quantity. Quantity is not a mere direct development from number, since a new conception, namely that of equality of units, enters as a distinctive factor which is not purely logical. It is true that equality for merely formal developments could be defined as a certain relation having the formal properties of symmetry and transitivity, and if to this conception is added the fundamental operation plus (+), definable as a certain relation having the formal properties commutative and associative, the whole system of quantitative science could be developed without recourse to any but pure mathematical principles. But even in this range of thought quantities of different types would need recognition. For example, given the notion of length as the first spatial quantity, a new quantity is derived by multiplying length by length, which is called area; here 'multiplied' need not be more specifically defined than a certain relation having the formal properties commutative and associative. Again where a quantum of space is divided by a quantum of time, we have velocity, and in this way a totally new type of quantity is constructed and we pass from geometry to kinematics. Another quantity called mass is such that when multi-

plied by velocity there is engendered the new quantity called momentum, and when multiplied by velocity squared, energy; and in the introduction of these new species of quantity we pass from kinematics to dynamics. This is the terminus on these lines of applied mathematics; and dynamics may be defined as the science that uses the three independently definable species of quantity time, space and mass. In every extension, then, of mathematics no new idea or mode of thought need accompany the work of the calculus. It is only when the formulae have to be applied to reality, and thus to be entertained categorically, that a process of thought other than merely mathematical enters in, and intuition is directed to what is given in some form of experience. The ideas which enter into the mathematical sciences thus constructed have a form which renders them amenable to purely logical processes of indefinite degrees of complexity; this distinguishes them from the non-mathematical or 'natural' sciences that introduce ideas dependent simply upon brute matter, unamenable to logical analysis, logic entering only in the application to these ideas of classification, and the principles of inductive inference.

§ 8. Having considered logic in its relation to the different sciences, we may now pass to a discussion of its more philosophical aspects. Logicians have been classified as nominalists, conceptualists, and realists or materialists, according as they think it worth while to discuss words, thoughts or things. Names that are apt to be understood as synonyms for these have been applied to different philosophical opinions; and this fact is indicative of the change which has occurred in



the course of the history of philosophy, where the ground has been shifted from ontology to psychology, and later from psychology to logic. To take realism first. It is the name given to the Platonic view which formed the basis of Aristotle's controversy with Plato. Plato in discussing the relation between the universal and the individual, attributes *real* existence in the truest or most ultimate sense to the universal, holding that the particular individual has reality only so far as it partakes of the nature of the universal, towards which it strives as the end (*ἐντελέκη*) of its existence. Aristotle, opposing this view, holds that the universal exists not apart from the particular but in it.

A new psychological significance came to be attached to the term Realism, when the question of reality was raised not about the *thing*, but about the possible *idea* of the thing, these two concepts being taken to be equivalent. The so-called nominalist school of philosophers maintained the psychological view that we had no idea corresponding to a general name, along with the ontological view according to which the particular individual or concrete alone existed, and no existence could be attributed to the universal; generality, for them, attached only to names in use, and had no objective application. On the psychological point at issue the opponents of this view have been known as conceptualists, and in maintaining their opposition were led to make a psychological distinction of great importance between images and ideas. In common with the nominalists, they held that images are necessarily concrete, particular or individual, but they maintained that we can also frame ideas which can properly be

called abstract or general. Both schools assumed that images were equivalent to or at least resembled perceptions, and further that the latter were obviously concrete and particular. Berkeley represents the nominalist school, and his subtle difference from Locke—who definitely held that we can frame general ideas, though with difficulty—comes out clearly when he disputes the possibility of a general idea of a triangle (instanced by Locke) which shall be neither equilateral nor isosceles nor scalene, and from which we can in thought abstract the shape from variations of colour. In my view Locke and Berkeley were both wrong, even where they agreed; inasmuch as neither images nor perceptions reflect the concreteness and particularity of the individual thing, which should be described as determinate, in contrast to the indeterminateness of the mental processes. In fact there has been a confusion in the description of our thoughts, images and percepts, between the distinction of the universal from the particular, and that of the indeterminate from the determinate. The modern term 'generic,' which has been applied to images, should be extended also to percepts, on the ground that they share with images the character of indeterminateness—a character which must be rigidly distinguished from general or universal as properly applied to ideas or concepts.

Nominalism has yet another meaning when applied as a special logical theory; in this sense it denotes the theory according to which the proposition is an indication of the names that have been arbitrarily chosen to denote things or classes of things, and predicates merely what follows from the consistent use of these names.

Propositions are thus used as mere formulae and repeated in thought when necessary, without demanding any consideration of their meaning; so that the only ultimate foundations or premisses of knowledge are definitions, no other propositions of the nature of axioms being required. This view still clings to some modern philosophical expositions of arithmetic and pure logic, and is rather subtly akin to the view that the first premisses for science are nothing but postulates or hypotheses which, if *consistently* held, lead to the discovery of truth.

As regards Conceptualism, it is doubtful whether, as applied to the work of such writers as Hamilton and Sigwart, it can be properly regarded as a distinctive logical theory. For the prominent use of the word concept and its associate judgment points not necessarily to any difference of logical theory between those who use these words, and those who prefer the words 'term' or 'name' and 'proposition,' but merely to the common recognition that thought has form as well as verbal expression. If, however, the conceptualist proceeds to limit the scope of logic to the consideration of the forms of thought alone, then he must maintain that the truth of a judgment is tested by the form that connects the content as conceived; and conceptualism becomes equivalent to formalism. The criterion for the formalist is indeed mere consistency or coherence in fact; that for the conceptualist proper, clearness or distinctness in thought. The latter is expressed negatively by Herbert Spencer: what is clearly not conceivable is false; positively by Descartes: what is clearly conceivable is true. It follows immediately from this view that truth concerns only conceived content;

so that the direct objects of thought are not things, but our ideas about things, and judgment contains no reference to things but only to adjectives. On this understanding, the conceptualist's view is that we can only deal with things as conceived, and that it is the mode under which we conceive them that determines the adjectives themselves and their relations as constituting the content of the judgment. In this way they are led to deny all relations as subsisting between things—a denial which is simply equivalent to denying the one supreme relation *otherness*; for *otherness* may be said to be the one determinable relation to which all specific relations stand as determinates. Hence it is enough for this school of philosophy to deny the single relation otherness, and in this denial to adopt the position of monism. The view, if carried out rigidly, goes beyond that of Spinoza, who asserted that thought was *other than* extension, and even that the one Substance had an infinity of other attributes, though not conceivable by us. It is an odd fact that Lotze, in particular, explicitly rejects relations only, as expressive of the nature of Reality; but in consistency he ought to have included in his rejection ordinary adjectives. From this point of view, the only kind of singular categorical judgment concerns Reality as a whole and not any one of its several separable parts: it predicates character of the indivisible *one*, not of this or that unit in the one. Individual units, in fact, are conceived as the result of the imposition of thought to which nothing in the *one* corresponds. Thus the Monist's first principle is to deny the Pluralist's fundamental assumption that the Real, as given to thought, is given as many and as such involves existential otherness.

The conceptualist's account of the character of the singular judgment leads to a similar account of that of the particular and the universal judgment. The view is consistently borne out by his interpretation of particulars as possible conjunctions, i.e. of adjectives that we *can* conjoin in conception; and of universals as necessary conjunctions, i.e. of adjectives that we *must* conjoin in conception. Symbolically: 'Some things that are  $p$  are  $q$ ' is to mean ' $p$  and  $q$  can be conjoined and can be disjoined'; 'Everything or nothing that is  $p$  is  $q$ ' is to mean ' $p$  and  $q$  must be conjoined or must be disjoined.' What is true in this view is that the operations *not*, *and*, *not-both*, *if*, *or*, are supplied by thought; and that nothing in the merely objective world manifests the mere absence of a character, or the mere indeterminateness of the alternative operation, or dependence as expressed by implication. These relations are not manifested *to* thought, but analytically or synthetically discovered or rather imposed *by* thought. The view is most strikingly expressed by Mr Bradley in his dictum: only if what is possible is necessitated will it be actualised; and again, only if what is necessary is possible will it be actualised.

From conceptualism we pass back again to realism in its new sense as applying to logic, and in this application it is usually denoted by the term materialism or empiricism. We are thus led back again to Venn, and less explicitly to Mill, who contrasts the formalism or conceptualism of Hamilton with his own logical standpoint. Taking empiricism to mean that all knowledge is obtained by experience alone (as Mill only seems to have held) the doctrine amounts to maintaining that all inference is ultimately of the nature of pure induction. But taking it to mean that no knowledge gained by

experience can be validly universalised (as Venn seems to hold) then the doctrine amounts to maintaining that no inference of the nature of pure induction is valid, and that hence only deduction is guaranteed by logic. In default of any explication of which of these two views is meant by empiricism or materialism, we can only conclude that the term stands for that department of logic that is concerned with an analysis of the process of induction. But here we must note that the distinction in character between induction and deduction is not properly expressed by the antithesis of matter and form; since the relations amongst premisses and conclusion <sup>a.</sup> which constitute the form of an inference hold for the validity of induction as for that of deduction; and conversely, reference to the matter of the propositions is required equally for the truth of a deductive inference as for that of an inductive inference. This obvious fact has been forgotten, owing to the great prominence given by inductive logicians to the treatment of the preliminary processes of observation, search, arrangement, comparison of material data, and the formation of formulae that shall hold for the facts collected, and the aid required by experimentation. In consequence, stress is laid on the securing of correctly described premisses in the case of induction; whereas in the case of deduction stress is laid only on securing validity for the form of inference.

§ 9. In conclusion I propose to enumerate the most important features in the treatment of logical theory to be developed in the course of this work:

(a) The epistemic aspect of thought is included within the province of logic, and contrasted with the constitutive aspect; the former is a recognition that



knowledge depends upon the variable conditions and capacities for its acquisition; the latter refers to the content of knowledge which has in itself a logically analysable form. Such fallacies as *petitio principii* really require reference to the epistemic aspect of thought, while fallacies of the strictly formal type refer exclusively to the constitutive aspect. Again the whole theory of modality which develops into probability is essentially epistemic, indicating as it does the relation of the content of the proposition to the thinker. Thus a distinction is clearly drawn between the proposition and the attitude of assertion or judgment; and while, on this view, the proposition is identifiable when in variable relations to different thinkers, the necessity is emphasised of conceiving the proposition in terms of assertion, the act of assertion being thus taken as the complete fact to be analysed and criticised. It is this intimate connection between the assertion and the proposition which gives meaning to the identification of the adjectives true and false with the imperatives 'to be accepted' and 'to be rejected.'

(b) The proposition itself, which is customarily resolved into subject and predicate, is more precisely analysed by showing that the substantive alone can function as subject, and the adjective as predicate, and that these stand to one another in the relation of characterisation: the substantive being that which is characterised, the predicate that which characterises. Since an appropriate adjective can be predicated of a subject belonging to any category, including adjective, relation and proposition, the subject as thus functioning becomes a quasi-substantive. The substantive proper seems to coincide with the category 'existent,' while

See p. 9

we do not usually  
class relation  
as 'adjective'?

This is incorrect, if what precedes is upheld, along with the view given on p. 9. If the substantive is defined by the way in which it functions in a possible proposition (etc.), then there is no such case of quasi-substantive: in 'Red' is an adjective

if any category other than substantive stands as subject its logical nature is not thereby altered, but rather the adjectives proper to it fall under correspondingly special sub-categories determined by the category to which the subject belongs.

(c) Adjectives are fundamentally distinguishable into determinables and determinates, the relation between which is primarily a matter of degree, a determinable being the extreme of indeterminateness under which adjectives of different degrees of determinateness are subsumed. The relation of a determinate to its determinable resembles that of an individual to a class, but differs in some important respects. For instance, taking any given determinate, there is only one determinable to which it can belong. Moreover any one determinable is a literal *summum genus* not subsumable under any higher genus; and the absolute determinate is a literal *infima species* under which no other determinate is subsumable.

(d) Relations are treated as a specific kind of adjective, and are called transitive adjectives in distinction from ordinary adjectives which are intransitive. The adjectival nature of relations is apt to be obscured by the inclusion under relative terms of what are merely substantives defined by relational characterisation. All that holds universally of adjectives, including the relation of determinates to their determinable, holds of relations as such.

(e) Under the head of induction, fundamentally different types are distinguished. First: the very elementary process of intuitive induction, which lies at the basis of the distinction between form and matter, and by which all the formal principles of logic are estab-

2. lished. Next: summary induction, more usually called perfect induction, which establishes conclusions of limited universality by means of mere enumeration. Such a summary universal stands as premiss for an unlimited universal conclusion, obtained by what is called *inductio per simplicem enumerationem*. What is specially important in my treatment is the function of summary induction in the specifically geometrical form of inference.
3. Thirdly: demonstrative induction, which employs no other principles than those which have been recognised in deduction. This species of induction is directly employed in inferring from a single experimental instance an unlimited universal; and it is this species of induction which gives the true form to the methods formulated by Mill and Bacon. Lastly we distinguish
4. induction proper, which is conceived as essentially problematic, and as thus re-introducing the epistemic aspect in the form of probability.

(f). The specific notion of cause as applying to events is distinguished from the generic notion of mere determination according to a universal formula. As specific, cause relates exclusively to states or conditions temporally alterable and also referable to place; and, in this application of the notion of determination, the effect and cause are homogeneous. Not only is the character of the effect regulated by that of the cause, but the date and place of the latter is determined by the date and place of the former. The universal positional relation, as it may be called, of cause to effect is that of contiguity, which is to be conceived in the form of the coincidence of the temporal or spatial boundary of that which constitutes the cause with that which constitutes the effect. This absolute contiguity disallows any gap

between the cause process and the effect process; so that contiguity is strictly defined as equivalent to continuity. This further implies that when, as is always permissible, we conceive a phase of the causal process as temporally or spatially separated from a phase of the effect process, we must also conceive of that which goes on in the interval bridging cause and effect to be part of one continuous process. This is possible because time and space are themselves continuous. Thus change and movement are connectionally continuous, in the special sense that the character manifested at one instant of time or at one point of space differs from that manifested at another instant of time or at another point of space, in a degree the smallness of which depends upon that of the temporal or spatial interval. Again superimposed upon the continuity of this process, there is a discontinuity of the second order, ultimately due to the discontinuous occupation of space by different kinds of matter.

× (g) The notions of cause and substance reciprocally imply one another, the latter being that which continues to exist and in which alterable states or conditions inhere. These alterable states constitute what may be called the *occurrent* or, in accordance with scholastic usage, the *occasional* causal factor. The occurrent is distinguished from and essentially connected with the *continuant* or the material factor in causation. The occurrent and continuant factors are thus united in our complete conception of substance, neither being conceivable apart from the other. This analysis gives meaning to the conception of the properties of the continuant, as potential causes which are actualised in accordance with unchanging rules by the relatively

incidental occurrences that come into being either from within or from without the continuant. In the former case the process is immanent, cause and effect being manifestations of the changeless nature of the continuant, and the temporal relation between cause and effect is here that of succession. In the latter case, the causality is transeunt, the patient being that whose state is determined, the agent being that whose alterable relation to the agent is determinative. In transeunt causality the temporal relation of cause to effect is literal simultaneity, and the critical instant at which the cause operates is that in which there is also literal geometrical contact of cause agent with effect patient. There are two fundamentally distinct types of transeunt causality.

1. In the one case no change of state in the agent accompanies the change of state in the patient, and we have action without any direct reaction; in the other
2. case change of state in the one directly entails change of state in the other of such a nature that the latter may be formulated as a function of the former, and here action always involves an assignable reaction. The latter case holds invariably of inter-physical causality, and again of inter-psychical causality within the sphere of a single individual's experience. But in physico-psychical causality, as also in psycho-physical causality, action never directly determines reaction, owing to the absolute disparateness between the physical and psychical in regard to the characters of the states which are predicable of the one and of the other. It is here where my treatment of logical questions transgresses into the domain of ontology; but it must be admitted that all logicians who treat these subjects inevitably transgress in the same manner.

(h) The position assumed by probability in logical discussion has always been dubious. On the one side the topic has been assumed to be the exclusive property of the mathematician, or rather more precisely, the arithmetician. On this view the quantity called probability is a mere abstract fraction, and the rules of probability are merely those of arithmetic. The fraction is, in short, the ratio of two numbers, the number holding for a species to that holding for its proximate genus, this ratio being necessarily a proper fraction, the limits of which are zero and unity. If this view were correct, there would be no separate topic to be called probability. A precisely reverse account of probability is that it is a measure of a certain psychological attitude of thought to which the most obvious names that could be given are belief or doubt, taken as subject to different degrees. On either of these two extreme views probability would have no particular connection with logic. The psychological account would be separated from logic, inasmuch as it would concern solely the causal explanation of different degrees of belief, and would thus give rise to no principle of rational criticism. The mere arithmetical account of probability ought in the first instance to be corrected by the recognition that the topic has its mental side. This correction requires that probability should not be expressed by a merely abstract fraction, but rather as a fraction of a certain mental quantity which may be called certainty. The psychological conditions of the variable degrees in which doubt may approximate to certainty are as such outside the province of logic; but when these various degrees are such as reason would dictate, we may speak of reasonable doubt as an assignable fraction of certitude,

*De Morgan*

thus bringing the subject into the sphere of logic. Further the quantity or degree called probability attaches exclusively to the proposition; not however to the proposition as such, but to the proposition regarded as based upon rationally certified knowledge acquired by any supposed thinker. The degree of probability is therefore referential to such knowledge, but is wholly independent of the individual thinker, being dependent solely on his rational nature, and the knowledge which he has rationally acquired.

*Part IV*

The whole development of this aspect of the subject is to be called formal probability, and constitutes the one subject of the fourth Part of this work. The treatment of probability there developed must be distinguished from that of informal probability, that is required in discussing the foundations of science as treated in my third Part; for there, while the logic of inductive inference is made to depend upon the principles of probability and not upon any big fact about nature, yet probability is only introduced on broad and indeterminately quantitative lines. This treatment leads to an attempted enumeration of broadly formulated criteria for the evaluation of the degrees of probability to be attached to the generalisations of inductive inference. These criteria are merely expressions of what is popularly felt, and their rational justification can only be represented as depending upon postulates: that is, speculations that are neither intuitively self-evident nor experientially verifiable, but merely demanded by reason in order to supply an incentive to the endeavour to systematise the world of reality and thus give to practical action an adequate prompting motive.

## CHAPTER I

### THE PROPOSITION

§ 1. A SYSTEMATIC treatment of logic must begin by regarding the proposition as the unit from which the whole body of logical principles may be developed. A proposition is that of which truth and falsity can be significantly predicated. Some logicians have taken the *judgment* as their central topic, and it will be necessary to examine the distinction between what I have called a proposition and what appears to be meant by a judgment. It has been very generally held that the proposition is the *verbal expression* of the judgment; this, however, seems to be an error, because such characterisations as true or false cannot be predicated of a mere verbal expression, for which appropriate adjectives would be 'obscure,' 'ungrammatical,' 'ambiguous,' etc. There appear then to be three notions which, though intimately connected, must be clearly distinguished: namely (1) what may be called the sentence; (2) the proposition; and (3) the judgment. The sentence may be summarily defined as the verbal expression of a judgment or of a proposition; it remains, therefore, to distinguish and interrelate the proposition and the judgment.

The natural use of the term judgment is to denote an act or attitude or process which may constitute an incident in the mental history of an individual. As so conceived, we should have further to distinguish the

changing phases of a process (which might alternately involve interrogation, doubt, tentative affirmation or negation) from the terminus of such process in which a final decision replaces the variations undergone during what is commonly called suspense of judgment. It would thus be more natural to speak of passing judgment upon a proposition proposed in thought than to identify judgment as such with the proposition. This more natural usage (which is that which I shall adopt) entails the necessity of recognising the distinction between various attitudes of thought on the one hand, and the object towards which that thought may be directed on the other; and even further, when necessary, of recognising the adoption of any of these alterable attitudes of thought as a datable occurrence within the total experience of some one individual thinker. There will thus be many fundamental attributes that must be predicated of the judgment upon a proposition different from, and often diametrically opposed to, those attributes that are to be predicated of the proposition itself.

In this account the judgment is the more comprehensive or concrete term, since when seriously treated it involves the two terms thinker and proposition and, in addition, the occurrent and alterable relation that may subsist between them. In thus drawing attention to mental process in my exposition of logical doctrine, I am taking what has been unfortunately termed a 'subjective' point of view. For the term 'subjective' should be substituted 'epistemic'; and in discarding the familiar antithesis *subjective* and *objective*, it is better for the purposes of Logic to substitute the antithesis *epistemic* and *constitutive*. The epistemic side of logical doctrine points

'epistemic'?

to the quite universally acknowledged kinship of Logic with Epistemology, and, in using this term in preference to subjective, we can avoid any confusion between what belongs to Psychology as opposed to what belongs to Logic. As to the term constitutive—a term for which philosophers are indebted to Kant—it has the force of 'objective' inasmuch as it points to the constitution of such an object of thought-construction as the proposition when treated independently of this or that thinker. I may anticipate what will be treated fully in the later part of logical doctrine, by pointing out that the distinction and connection between the epistemic and constitutive sides of logical problems plays an important part in the theory of Probability; and, in my view, it ought to assume the same importance throughout the whole of the study of Logic.

Now, as regards the relation of the proposition to any such act as may be called judgment, my special contention is that the proposition cannot be usefully defined in isolation, but only in connection with some such attitude or act of thought; and I prefer to take the notion of *asserting* as central amongst these variations of attitude—which will therefore be spoken of as variations in the assertive attitude. I shall also maintain that the fundamental adjectives true and false which are (perhaps universally) predicated of mere propositions as such, derive their significance from the fact that the proposition is not so to speak a self-subsistent entity, but only a factor in the concrete act of judgment. Thus, though we may predicate of a certain proposition—say 'matter exists'—that it is true or that it is false, what this ultimately means is, that any and every thinker who might

here, as appears to  
be "assertion" in  
a very wide sense  
of the word



at any time assert the proposition would be either exempt or not exempt from error. In other words, the criticism which reason may offer is directed—not to the proposition—but to the *asserting* of the proposition; and hence the customary expression that such and such a proposition is false merely means that anyone's assertion of the proposition would be erroneous. The equivalence of these two forms of criticism follows from the fundamental principle that an attitude of assertion is to be approved or condemned in total independence of the person asserting or of the time of his assertion, and in exclusive dependence upon the content of his assertion. This fundamental principle of Logic will come up for detailed treatment when the so-called Laws of Thought are explicitly discussed. In order to mark the important distinction, and at the same time the close connection, between the proposition and the act of assertion, I propose to take the term 'assertum' as a synonym for 'proposition' when such terminology may seem convenient. Thus, the assertum will coincide, not exactly with what *has been* asserted, but with what is in its nature assertible.

§ 2. Many philosophers have used the term belief in its various phases as a substitute either for judgment or for assertion; in fact, when the mental aspect of any problem assumes special prominence, the term belief as applied to the proposition is more naturally suggested than any other. While the object of belief is always a proposition, the proposition may be merely entertained in thought for future consideration, either without being believed, or in a more or less specific attitude opposed to belief, such as disbelief or doubt. To doubt

a proposition implies that we neither believe nor disbelieve it, while belief and disbelief as opposed to doubt have in common the mental characteristic of assurance. Thus there are three opposed attitudes towards a proposition, included in the distinction between assurance and doubt;—the former of which may be either (assured) belief or (assured) disbelief, and the latter of which appears further to be susceptible of varying felt degrees. The close association amongst all the terms here introduced brings into obvious prominence the mental side, which such terms as judgment or assertion seem hardly to emphasise. It would however, I think, be found that there is in reality no relevant distinction between the implications of the two terms 'judgment' and 'belief.' Those logicians who have spoken exclusively of judgment, conception, reasoning, etc., have had in view more complicated processes, the products of which have been explicitly formulated; while those who have used belief and cognate terms have included more primitive and simple processes, the products of which may not have been explicitly formulated. Since the traditional logic has treated *only* the more developed processes, the term judgment and its associates is perhaps preferable for this somewhat limited view of the scope of Logic, while the use of the term belief—which must certainly be understood to include the higher as well as the lower processes—points to a wider conception of the province of Logic. To put the matter shortly, I hold it to be of fundamental importance to insist that there is *some* factor common to the lower and higher stages, and that this common factor, to which the name belief has been given, is necessarily directed to what

in Logic is called a proposition<sup>1</sup>. Assertion, in the sense here adopted, is to be understood to involve belief, and may be defined as equivalent to *conscious belief*. This definition restricts the term in two ways: in that, firstly, to assert does not merely mean to *utter* (without belief); and secondly, merely to believe *unconsciously* is not to assert.

§ 3. In speaking of variations of attitude towards the proposition, an assumption is involved that there is a single entity called the proposition that is the same whatever may be the attitude adopted towards it. Ordinary language supplies us with names for such different attitudes along with cognate names for the proposition: thus we associate 'to assume' with 'an assumption'; 'to suppose' with 'a supposition'; 'to propose' with 'a proposition'; 'to postulate' with 'a postulate'; 'to presume' with 'a presumption'; etc.<sup>2</sup> Consider the two verbs 'to assume' and 'to presume.' It will be acknowledged that these denote attitudes between which some subtle distinction may be under-

<sup>1</sup> Readers of Psychology should be warned that, when psychologists contrast 'imagination' with 'belief,' each term indicates an attitude to a proposition; while, when they contrast 'imagination' with 'perception,' the processes to which they refer do not involve any attitude towards a proposition. There is no common element of meaning in these two applications of the word 'imagination.'

<sup>2</sup> In further illustration of this point we may select certain prominent logical terms such as hypothesis, postulate, axiom. Each of these terms indicates the peculiar attitude *to be* assumed towards the proposition in question by *any* thinker: thus a hypothesis stands for a proposition which awaits further scientific investigation before being finally accepted or rejected; a postulate stands for a proposition which cannot be brought to the test of experience, but the truth of which is *demanded* by the thinker; and an axiom is a proposition the truth of which is self-evident to the thinker.

stood; and thus it might appear that in correspondence with this distinction there must be a similar subtle distinction between an assumption and a presumption. Unfortunately substantival words such as these are apt to suggest a difference in nature between that which in the one case is presumed and in the other assumed; but this suggestion must be rejected, and it must be maintained on the contrary that the content of a proposition preserves its identity unmodified, independently of all variations of assertive attitude and of personal and temporal reference. This independence holds also in regard to what has been termed 'logical' in contrast with 'psychological' assertion. The phrase *logically asserted*, applied to this or that proposition, is only metaphorically legitimate, and literally equivalent to 'asserted on purely rational grounds by any or all rational persons.' In other words, the predicate 'asserted' conveys no meaning when taken apart from a person asserting.

Adopting as we do the general view that no logical treatment is finally sound which does not take account of the mental attitude in thought, it follows that the fundamental terms 'true' and 'false' can only derive their meaning from the point of view of criticising a certain possible mental attitude. We are thus bound to distinguish the object of this attitude (the assertum) from the attitude itself which may vary independently of the object; but we can only avoid contradiction or vagueness if, while permitting ourselves to distinguish between the attitude and its object, we at the same time refuse to separate them. We may further explain the adjectives 'true' and 'false' so as to bring out what



characterises logic in contrast with—or rather in its relation to—psychology: namely that logic formulates standards or imperatives which as such have no significance except as imposed upon mental acts. Thus we may say that the application of the adjectives true and false coincides with the application of the imperatives 'to be accepted' and 'to be rejected' respectively. We may add that these imperatives are imposed by the thinker—in the exercise of his reason—upon himself. In maintaining this coincidence between the two imperatives on the one hand and the two adjectives (true and false) on the other, it must not be taken that we are able thus to *define* the adjectives true and false. On the contrary, we are forced to insist that they are indefinable. We are only indicating that a reference to mental attitude is presupposed when Logic recognises the distinction between true and false in its formulation of standards for testing the correctness of a judgment or assertion.

§ 4. So far we have taken the proposition *as a unit* of which the adjectives true and false may be predicated. Before proceeding to *analyse* the proposition into its component parts, a word must be said in regard to the relation of logic to universal grammar, and in particular the relation between grammatical and logical analysis. Properly speaking, grammatical analysis cannot be regarded as dealing merely with words and their combinations. The understanding of the grammatical structure of a sentence—which includes such relations as those of subject to predicate, and of subordinate to co-ordinate clauses—requires us to penetrate below the mere verbal construction and to consider the formal structure of

thought. Hence, on the one hand, grammar cannot be intelligently studied unless it is treated as a department of logic; and, on the other hand, logic cannot proceed without such a preliminary account of linguistic structure as is commonly relegated to grammar. In short, universal grammar (as it is called) must be subsumed under Logic. On this view, a slight alteration in grammatical nomenclature will be required, whereby, for the usual names of the parts of speech, we substitute substantive-word or substantive-phrase, adjective-word or adjective-phrase, preposition-word or phrase, etc., reserving the terms substantive, adjective, preposition, etc., for the different kinds of entity to which the several parts of speech correspond.

§ 5. To turn now to the analysis of the proposition. We find that in every proposition we are determining *in* thought the character of an object presented *to* thought to be thus determined. In the most fundamental sense, then, we may speak of a determinandum and a determinans: the determinandum is defined as what is presented *to be* determined or *characterised by* thought or cognition; the determinans as what *does* characterise or determine in thought that which is given to be determined. We shall regard the substantive (used in its widest grammatical sense) as the determinandum, and the adjective as the determinans. Neither of these terms can be defined except in their relation to one another as each functions in a *possible* proposition. As it has frequently been said, the proposition is *par excellence* the unit of thought. This dictum means that the logical nature of any components into which we may analyse the proposition can only be defined by the mode in

See Morris' book  
on Metaphysics  
p. 52 ff.

But see *Logic*  
p. xxiv.

But the  
only to the  
logical unit  
of each.  
see p. 11.

which they enter into relation within it. For example, when I use *determinandum* for the substantive and *determinans* for the adjective, I am only defining the one in terms of the other, inasmuch as the common factor 'determine' is contained in both. This account goes beyond that which has become commonplace among many philosophers, namely, that the subject of a proposition is ultimately something which cannot be defined in the way in which a predicate or adjective can be defined; for to this we have to add that the predicate of a proposition is ultimately something which cannot be defined in the way in which a subject or substantive can be defined. These two statements present the natures of subject and predicate purely negatively, the positive element being supplied by the terms 'determinans' and 'determinandum.'

We have now to examine the nature of the connection involved in every case where adjective and substantive are joined; for example 'a cold sensation,' 'a tall man.' In order to understand the verbal juxtaposition of substantive and adjective, we must recognise a latent element of form in this construct, which differentiates it from other constructs—which also are necessarily expressed by a juxtaposition of words. This element of form constitutes what I shall call the *characterising tie*. The general term 'tie' is used to denote what is not a component of a construct, but is involved in understanding the specific form of unity that gives significance to the construct; and the specific term 'characterising tie' denotes what is involved in understanding the junction of substantive with adjective. The invariable verbal expression for the characterising tie is the verb

'to be' in one or other of its different modes. To think of 'a tall man' or of 'a cold sensation' is to think of 'a man as being tall,' 'a sensation as being cold.' Here the word 'being' expresses the characterising tie, and the fact that in some cases the word may be omitted is further evidence that the tie is not an additional component in the construct, but a mere formal element, indicating the connection of substantive to adjective. This is its peculiar and sole function; and, as the expression of the unique connection that subsists between substantive and adjective, it is entirely unmodifiable.

The distinction and connection between substantive and adjective correspond to—and, in my view, explain—the distinction and connection between particular and universal<sup>1</sup>. Ultimately a universal means an adjective that may characterise a particular, and a particular means a substantive that may be characterised by a universal. The terms particular (or substantive) and universal (or adjective) cannot be defined as functioning in isolation, but only as they enter into union with one another. There is some danger of confusing two different uses of the verb 'to characterise,' which may be partly responsible for the historical dispute concerning the relation of particular to universal. Primarily the term 'characterise' should be used to connect substantive with adjective in the form 'such and such a quality or adjective characterises such and such an object or substantive.' On the other hand, in the phrase 'the thinker characterises such or such an object,' characterises means

<sup>1</sup> Here the terms particular and universal are used in the sense current in philosophy, and not in their familiar application in elementary logic, where they stand for sub-divisions of the proposition.

but cf. p. 2, 10  
in page

See pp. 212, 213

What does Johnson mean by a 'construct'?

See Whorw's  
conjunction of these  
words, in the  
structure of the sentence  
p. 45 ff (2nd Series)

Understanding?

x 'cognitively determines the character of.' Owing to this elliptical use of the term, the particular has been conceived of as 'an uncharacterised object,' and this would mean literally 'an object without any character'; but since actually every object must have character, the only proper meaning for the phrase 'uncharacterised object' is 'an object whose character has not been cognitively determined.' If then the term 'exist' may be predicated equally of a universal as of a particular, then we may agree with the Aristotelian dictum that the universal exists, not apart from, but in the particular; and by this is meant that the adjective exists, not apart from, but as characterising its substantive; to which must be added that the substantive exists, not apart from, but as characterised by its adjective. Now in thought the substantive and the adjective may be said to be separately and independently represented; hence thinking effects a severance between the adjective and the substantive, these being reunited in the *asserted* proposition—not only by the characterising tie, but *also* by what we may call the *assertive* tie. The blending of the assertive with the characterising tie is expressed in language by the transition from the participial, subordinate, or relative clause, to the finite or declaratory form of the principal verb. Thus in passing from 'a child fearing a dog' to 'a child fears a dog,' the characterising tie joins the same elements, in the same way, in both cases; but is, in the latter, blended with the assertive tie. That the ties are thus blended is further shown by the modifications 'is-not,' 'may be,' 'must be,' by which the verb 'to be' is inflected in order to indicate variations in the assertive attitude while the character-

of Meinong's 'annexation' or 'determination'  
Lambert's 'propositional complex'.

ising relation remains unchanged. The copula '*is*' of traditional logic is thus seen to be a blend of the characterising with the assertive tie.

§ 6. We must now criticise a view, explicitly opposed to our own, as to the nature of the copula *is*. There has been for a long period an assumption that the proposition in some way or other asserts the relation of identity. This relation of identity, it is admitted, is not one of complete or absolute identity, but involves also a relation of difference: thus the proposition 'Socrates is mortal' is transformed into 'Socrates is a mortal being'—where 'Socrates' and 'a mortal being' are affirmed to be identical in denotation but different in connotation. Have logicians quite recognised the extreme elaborateness of this verbal transformation? The adjective 'mortal' has first to be turned into a substantive in using the word 'a mortal being'; secondly, the indefinite article has to be introduced, since it is clear that Socrates is not identical with every mortal; thirdly, the indefinite article has to be carefully defined as meaning *one or other*; fourthly, the relation of the adjective 'mortal' to the substantive 'being' which it characterises still remains to be elucidated; fifthly, another adjective (a relational adjective) namely *identical* is introduced in the compound phrase 'is identical with.' The proposition finally becomes: 'Socrates is identical with one or other being that is mortal.' Here the two adjectives 'mortal' and 'identical with' are each introduced after *is*. Now, if 'is identical with' is to be substituted for *is* in each case, then we shall arrive at an infinite regress. Thus, in the first place, 'Socrates *is* identical with *X*' (say) must be rendered 'Socrates

*is identical with* a being that is identical with *X*' where the force of *is* still remains unexplained. And in the second place 'one or other being that *is* mortal' must be rendered 'one or other being that *is identical with* a mortal,' where again *is* still remains to be explained. In each case, if an infinite regress is to be avoided, the word *is* that remains must be interpreted as representing the unique mode in which the fundamentally distinct categories substantive and adjective are joined.

§ 7. Having so far considered the proposition in its mental or *subjective* aspect, we have next to examine it in what may be called its *objective* aspect. Whereas a proposition is related subjectively to *assertion*, we shall find that it is related objectively to *fact*<sup>1</sup>. Our conclusion, briefly expressed, is that any proposition *characterises* some fact, so that the relation of proposition to fact is the same as that of adjective to substantive. Bradley has represented a proposition as ultimately an adjective characterising Reality, and Bosanquet as an adjective characterising that fragment of Reality with which we are in immediate contact. In adopting the principle that a proposition may be said, in general, to characterise a fact, I am including with some modification what is common to these two points of view.

a. One parallel that can be drawn between the relation of an adjective to a substantive and that of a proposition to a fact is that, corresponding to a single given substantive, there are an indefinite number of adjectives which are truly predicable of it, just as there are many different propositions which truly characterise any given

<sup>1</sup> Otherwise expressed: The proposition, subjectively regarded, is an *assertibile*; objectively regarded, a *possibile*.

( Subjective ) Assertion ← Proposition → Fact ( objective )

fact. Thus we do not say that corresponding to a single fact there is a single proposition, but on the contrary, corresponding to a single fact there is an indefinite number of distinct propositions. Again, just as amongst the adjectives which can be truly predicated as characterising a given substantive, some are related to others as relatively more determinate; so, amongst the several propositions which truly characterise a single fact, some characterise it more determinately and thus imply those which characterise the same fact less determinately. We may therefore regard the process of development in thought as starting from a fact given to be characterised, and proceeding from a less to a continually more determinate characterisation.

Again there is an exact parallel between the relation of contradiction or contrariety amongst adjectives that could be predicated of a given substantive, and amongst propositions which could be formulated as characterising a given fact. Thus the impossibility of predicating certain pairs of adjectives of the *same substantive* involves the same principle as the impossibility of characterising the *same fact* by certain pairs of propositions: such pairs of adjectives and propositions are *incompatible*, and this relation of *incompatibility* lies at the root of the notion of contradiction. We may illustrate the relation of incompatibility amongst adjectives by *red* and *green* regarded as characterising the same patch. It is upon this relation of incompatibility that the idea of the contradictory *not-red* depends; for *not-red* means some adjective incompatible with *red*, and predicates indeterminately what is predicated determinately by *green*, or by *blue*, or by *yellow*, etc. Amongst propositions

the relation of incompatibility may be illustrated by 'Every  $p$  is  $qu$ ' and 'Every  $pv$  is non- $q$ ,' which are more determinate forms of the pair of contradictory propositions 'Every  $p$  is  $q$ ' and 'Some  $p$  is non- $q$ .' These latter derive their significance as mutually contradictory from the principle that the actual fact must be such that it could be characterised *either* by such a relatively determinate proposition as 'Every  $p$  is  $qu$ ' *or* by such a relatively determinate proposition as 'Every  $pv$  is non- $q$ .'

This account of the relation of contradiction as ultimately derived from that of incompatibility or contrariety (whether applied to adjectives regarded as characterising substantives or to propositions regarded as characterising facts) brings out in another aspect the principle that any given substantive or any given fact may be truly characterised by a more or by a less determinate adjective or proposition: a topic which will be further developed in later chapters.

The above logical exposition of the nature of a proposition leads to a consideration of the philosophical problem of the relation of thought to reality in one of its aspects. It is at the present day agreed that this relation cannot be taken to be identity, and the notion of correspondence has been put forward in its place. The above account enables us to give a more definite exposition of what more precisely this so-called correspondence entails: the truth of a judgment (expressed in a proposition) may be said to mean that the proposition is in accordance with a certain fact, while any proposition whose falsity would necessarily follow from the truth of the former is in discordance with that fact.

In this way the somewhat vague conception of the correspondence between thought and reality is replaced by the relation of accordance with a certain fact attributed to the true proposition, and of discordance with the same fact attributed to the associated false proposition.



## CHAPTER II

## THE PRIMITIVE PROPOSITION

§ 1. THE form of proposition which appears to be psychologically prior even to the most elementary proposition that can be explicitly analysed is the exclamatory or impersonal. Propositions of this kind, which are more or less unformulated and which may be taken to indicate the early stages in a developing process, will here be called *primitive*. The most formless of such primitive propositions is the exclamatory assertion illustrated by such an utterance as 'Lightning!' This appears to contain only a characterising adjective without any assigned subject which is so characterised. Now it is true that any proposition can be regarded as a characterisation of the universe of reality regarded as a sort of unitary whole; but this way of conceiving the nature of the proposition in general, must be also associated with the possibility of using adjectives as characterising a part rather than merely the whole of reality; and certainly the case here is one in which we are bound to recognise the lightning as having, so to speak, an assignable place within the universe, and not merely as an adjective attached to the universe as a whole. The lightning as an actual occurrence must occupy a determinate position, in reference both to time and to space; but it is obvious that no reference to such determinate position is itself contained in the merely exclamatory assertion. Any implicit reference to place

## THE PRIMITIVE PROPOSITION

or time can only be rendered explicit when the judgment has been further developed; in the undeveloped judgment the reference is indeterminate, and any judgment which might be developed from this primitive form would assert what was unasserted in the original. In logical analysis it is of the utmost importance to avoid putting into an assertion what further development of the percipient's thought might elicit on the basis of the original.

We ask then, how such judgment in its most primitive and undeveloped form can be conceived as referring to a subject when its verbal expression includes no such reference? Now we may speak of the presented occasions or occurrences that give rise to such incompletely formulated judgments as *manifestations of reality*. The exclamatory judgment 'Lightning' may thus be rendered formally complete by taking as subject term 'a manifestation of reality.' Here I do not propose to take simply as the equivalent of the exclamatory judgment 'Reality is being manifested in the lightning,' but rather 'A *particular portion of reality* manifests the character (indicated by the adjectival import of the word) lightning.' In short, what is asserted by the percipient is 'a manifestation of lightning'.<sup>1</sup> This phrase for representing the assertum contains of course the characterising tie but not the assertive tie. The assertive tie may be introduced by employing the form: 'There is a manifestation of lightning,' which raises the interesting problem as to the significance of the word

<sup>1</sup> In grammatical phraseology, the expression 'manifestation of reality' illustrates the *subjective* genitive, while 'manifestation of lightning' illustrates the *objective* genitive.

'there.' Like many other words in current language it is used here in a metaphorical, or perhaps rather in a general or abstract sense. Literally 'there' means 'in that place,' so that in its original significance it involves the demonstrative article, and furthermore—which is the new matter of interest—a reference to position in space. Moreover the tense of the verb *is* points to the present time. If these references were developed still more precisely, the assertion would become: 'There and now—in that place and at this time—is a manifestation of lightning.' What remains as the significant element in the word 'there is,' in the absence of any *definite* reference to position in time or space, must be an *indefinite* reference to position in time and space. Otherwise the exclamatory assertion can only be expressed by omitting the word 'there' altogether, and the assertion to which we are reduced—when the subject implicit in the exclamation is made explicit—becomes, as above, '*A* manifestation of lightning<sup>1</sup>.'

§ 2. The phrase 'there is' points to an important presupposition underlying the possibility of this most primitive form of perceptual judgment: namely, that things should be presented apart or in separation in order that any characterising judgment may be directed now to one and then again to another. Thus separation of presentment is a presupposition of cognition or judgment. Here I use the word 'presentment' not as equivalent to cognition, but as something presupposed in all—even the most primitive—acts of cognition. The

<sup>1</sup> As an illustration of how words lose their philological origin and become merely metaphorical, consider the expressions: 'There is a God,' 'There is an integer between 5 and 7.'

word 'present'—with the accent on the second syllable<sup>1</sup>—is in English equivalent to 'give'; in this sense a *presentation* is equivalent to a *datum*—where by 'datum' is meant not a piece of given knowledge, but a piece of given reality that is to be characterised in knowledge. Thus the presentation or the datum is what I have otherwise called the *determinandum*—that which is given or presented to thought to be thought about. This expresses briefly, the meaning of the primitive 'this.' The 'this' as thus defined is not rich in predicates and adjectives, but at the same time it cannot be said to be empty of adjectives or predicates, because, in the meaning of thisness, abstraction is made from all predicates or adjectives. But the 'this' cannot be explicated apart from an implicit reference to the 'that,' in the sense that the 'this' must be for the percipient presented in separation from the 'that': one *determinandum* is *one* to which *its own* adjectives may be assigned, just because *the other* must be presented in separation or apart from *the one*, before the most primitive form of articulate judgment is possible<sup>2</sup>. Briefly separateness is *before* relating; more specifically, it is the presupposition which makes it possible in more highly developed perception to define the relations (temporal or spatial) between those things which are first presented merely as separate.

<sup>1</sup> When accented on the *first* syllable, its meaning combines a reference both to space and to time; so that the word *presentation* contains in its *meaning* the three factors in our analysis, viz. the given, the here, and the now.

<sup>2</sup> It is here presumed that such mental processes as sense differentiation, etc., in which the experient is merely passive or recipient, must have been developed prior to the exercise of judgment, to furnish the material upon which the activity of thought can operate.



It is in this quite ultimate sense that I demur to Mr Bradley's dictum: 'distinction implies difference.' His dictum means, as far as I understand, what in my own terminology I should express by the phrase: 'otherness presupposes comparison' (the comparison, in particular, in which the relation of difference is asserted). Now in my view this dictum is exactly wrong: the assertion of 'otherness' does not presuppose or require a previous assertion of any relation of agreement or of difference. It does not even presuppose the possibility of asserting in the future any *particular* relation of agreement or of difference. The first important relation which will be elicited from otherness is, in fact, not any relation of agreement or difference at all, but a temporal or spatial relation; and thus the primitive assertion of otherness is only occasioned and rendered possible from the fact of separateness in presentation. When presentations are separate, then we can count one, two, three; further, we can connect them by temporal relations such as before and after, or by spatial relations such as above or below; and finally by relations of comparison such as like or unlike. These examples indicate my view of the quite primary nature of separateness of presentment, since it is for me the pre-requisite for *all* those acts of connecting with which logic or philosophy—and we may add psychology—is throughout concerned. In illustration, I have briefly referred only to relations of number, relations of time and space, and lastly to relations of comparison in a quite general sense.

¶ Summarising this attempt to indicate the precise logical character of such primitive judgments as the exclamatory or impersonal, and their relation to more

highly developed judgments: we have found that to assert 'Lightning!' is to characterise, not reality as a whole, but a separate part of reality—to use an inadequate expression—and that the possibility for this primitive assertion to develop into higher, more inter-related forms of judgment, is wholly dependent on the attribution of an adjective to a part of reality presented in separation from other presentables.

For the purpose of further elucidation we may bring two or three assertions into connection with one another, which might be briefly formulated thus: 'Lightning now!' 'Lightning again!' 'Thunder then!' The first two judgments when connected, involve two manifestations of the same character denominated lightning, which are *two* because they have been *separately* presented. The use of the terms 'now' and 'then' does not necessarily presuppose a developed system of temporal relations; but they indicate at least the possibility of defining relations in time between separately presented manifestations. Again the exclamation 'Thunder!' when taken in connection with the exclamation 'Lightning!' already presupposes—not only that the manifestations are given somehow in separation—but further that the percipient has characterised the separated manifestations by different adjectives. I will not here discuss whether these *manifestations* (as I have called them) are, in their primitive recognition, merely the individual's sense-experiences of sound and light, or whether from first to last they are something other than sense-experiences. In either case our logical point will be the same, when it is agreed that they are given separately, and that their separate presentment is the

precondition for any further development of thought or of perception. The view put forward here is so far equivalent to Kant's in that I regard space and time as the conditions of the otherness of sense-experiences upon which the possibility of cognising *determinate* spatial and temporal relations depends, and that this characteristic of space and time is what constitutes Sense-Experience into a manifold, i.e. a plurality of experiences, which we can proceed to count as many only because of their separate presentment.

Taking more elaborate examples of these primitive forms of perceptual judgment: 'This is a flash of lightning,' 'This (same) flash of lightning is brighter than that (other),' 'This (same) flash appeared before that clap of thunder'; we note that in the predesignations 'this' and 'that' the percipient has passed beyond the indefinite article 'a,' and has identified a certain manifestation as that of which more than one characterisation can be predicated—e.g. 'lightning' and 'brighter than that.' It is this identification which gives to the articles 'this' and 'that' a significance which may be called *referential*, to be distinguished from their use as demonstratives; and in this alternation between the demonstrative and the referential usage, we can trace, I think, the very primitive way in which thought develops: first, in fixing attention upon a phenomenon by pointing to its position; and, next, in identifying it as the same in character when it is changing its spatial relations. All that is theoretically required for identification is the retention—or rather the detention—of our cognition or judgment upon a certain manifestation; but, when attributing *different* qualities or relations to what

continues to function as the *same* logical subject, we are assisted by the temporal continuance of a phenomenon, either with unaltered quality, or in an unaltered position, or in a *continuously* changing position, etc. Thus the changes involving variations in space, time, and quality amongst different manifestations of the same phenomenon constitute the groundwork upon which the several judgments of relation are built.

In asserting 'There was a flash of lightning that was very brilliant and that preceded a clap of thunder' we are grasping the identity of a certain manifestation, thus used in two propositions, of which one predicates a relation in time to a clap of thunder, and the other, a quality characterising the flash itself. Any such connected judgment contains implicitly the relation of identity, in that the manifestation is maintained as an object in thought, while we form two judgments with respect to it. It is only, in short, in the act of joining two different characterisations that any meaning for identity can be found. In an elementary judgment which predicates only *one* adjective, no scope or significance for the notion of identifying a subject as such can be afforded. Thus the three factors in the thinking process which the 'this' reveals are: (1) the given—which is equivalent to the 'it' in 'It lightens!' (2) the demonstrative—which, by indicating spatial position, helps towards unique identification, (3) the referential—which marks the achievement of this process of identification. As will be seen from the discussion in a subsequent chapter, these three elements of significance in the 'this' bring it into line with the proper name.

## X CHAPTER III

## COMPOUND PROPOSITIONS

§ 1. HAVING examined the proposition in its more philosophical aspects, and in particular from the point of view of its analysis, the present chapter will be mainly devoted to a strictly formal account of the proposition, and will be entirely concerned with the synthesis of propositions considered apart from their analysis. The chapter is intended to supply a general introduction to the fundamental principles of Formal Logic; and formulae will first be laid down without any attempt at criticism or justification—which will be reserved for subsequent discussion. For this purpose we begin by considering the different ways in which a new proposition may be constructed out of one or more given propositions.

In the first place, given a *single* proposition, we may construct its negative—expressed by the prefix *not*—*not-p* being taken as equivalent to *p*-false. Next, we consider the construction of a proposition out of two or more given propositions. The proposition thus constructed will be called *compound*, and the component propositions out of which it is constructed, may be called *simple*, relatively to the compound, although they need not be in any absolute sense simple. The prefix *not* may be attached, not only to any simple proposition, but also to a compound proposition, any of whose components again may be negative.

The different forms that may be assumed by compound propositions are indicated by different *conjunctions*. A proposition in whose construction the only formal elements involved are negation and the logical conjunctions is called a *Conjunctive Function* of its component propositions. The term conjunctive function must be understood to include functions in which negation *or* any one or more of the logical conjunctions is absent. We have to point out that the compound proposition is to be regarded, not as a mere plurality of propositions, but as a *single* proposition, of which truth or falsity can be significantly predicated irrespectively of the truth or falsity of any of its several components. Furthermore, the *meaning* of each of the component propositions must be understood to be assignable irrespectively of the compound into which it enters, so that the meaning which it is understood to convey when considered in isolation is unaffected by the mode in which it is combined with other propositions.

§ 2. We will proceed to enumerate the several modes of logical conjunction by which a compound proposition may be constructed out of two component propositions, say *p* and *q*. Of all such modes of conjunction, the most fundamental is that expressed by the word *and*: this mode will be called par excellence *conjunctive*, and the components thus joined will be called *conjuncts*. Thus the compound propositions—

(a) '*p* and *q*,' (b) '*p* and not-*q*,' (c) 'not-*p* and *q*,' (d) 'not-*p* and not-*q*,' are the conjunctive functions of the conjuncts *p*, *q*; *p*, not-*q*; not-*p*, *q*; not-*p*, not-*q*; respectively. There are thus four distinct conjunctive forms of proposition involving the two propositions *p*, *q*, taken positively or negatively.

The significance of the conjunctive *and* will be best understood in the first instance, by contrasting it with the enumerative *and*. For example, we use the merely enumerative *and* when we speak of constructing any compound proposition out of the components *p* and *q*. Here we are not specifying any mode in which *p* and *q* are to be combined so as to constitute one form of unity rather than another; we are treating the components (so to speak) severally, not combinatorially. In other words, the enumeration—*p* and *q*—yields two propositions, the enumeration—*p* and *q* and *r*—yields three propositions, etc.; but the conjunctive '*p* and *q*,' or the conjunctive '*p* and *q* and *r*' etc., yields one proposition. Again, of the enumerated propositions—*p* and *q* and *r* and...—some may be true and others false; but the conjunctive proposition '*p* and *q* and *r* and...' must be either definitively true or definitively false. Thus in *conjoining* two or more propositions we are realising, not merely the force of each considered separately, but their joint force. The difference is conclusively proved from the consideration that we may infer from the conjunctive proposition '*p* and *q*' a set of propositions none of which could be inferred from *p* alone or *q* alone. The same holds, of course, where three or more conjuncts are involved: thus, with *p*, *q*, *r*, as components, seven distinct groups<sup>1</sup> of propositions are generated: viz. the three groups consisting of propositions implied by *p*, by *q*, by *r* respectively; the three groups consisting of propositions implied by '*p* and *q*,' by '*p* and *r*,' by '*q* and *r*' respectively; and lastly, the group consisting of propositions implied by '*p* and *q* and *r*.'

<sup>1</sup> The term *group* is here used in its precise mathematical significance.

§ 3. In our first presentation of formal principles we shall introduce certain familiarly understood notions, such as equivalence, inference etc., without any attempt at showing how some of them might be defined in terms of others. The same plan will be adopted in regard to the question of the demonstrability of the formal principles themselves; these will be put forward as familiarly acceptable, without any attempt at showing how some of them might be proved by means of others. Ultimately, certain *notions* must be taken as intelligible without definition, and certain *propositions* must be taken as assertible without demonstration. All other notions (intrinsically logical) will have to be *defined* as dependent upon those that have been put forward without definition; and all other propositions (intrinsically logical) will have to be *demonstrated* as dependent upon those that have been put forward without demonstration. But we shall not, in our first outline, raise the question of the dependence or independence of the notions and propositions laid down.

Thus the formal law which holds of Negation is called the Law of Double Negation: viz. not-not-*p*  $\equiv$  *p*.

§ 4. We now lay down the formal laws which hold of compound propositions constructed by means of the conjunction *and*. They are as follows:

#### *Laws of Conjunctive Propositions*

1. The Reiterative Law:

$$p \text{ and } p \equiv p.$$

2. The Commutative Law:

$$p \text{ and } q \equiv q \text{ and } p.$$

3. The Associative Law:

$$(p \text{ and } q) \text{ and } r \equiv p \text{ and } (q \text{ and } r).$$

Here the notion of *equivalence* (expressed by the shorthand symbol  $\equiv$ ) is taken as ultimate and therefore as not requiring to be defined. These laws and similar formal principles are apt to be condemned as trivial. Their significance will be best appreciated by reverting to the distinction between the mental acts of assertion and progression in thought on the one hand, and the propositions to which thought is directed on the other. Thus the laws above formulated indicate, in general, equivalence as regards the propositions asserted, in spite of variations in the modes in which they come before thought. Thus the content of what is asserted is not affected, firstly, by any *re-assertion*; nor, secondly, by any different *order* amongst assertions; nor, thirdly, by any different *grouping* of the assertions.

§ 5. Having considered the *Conjunctive* form of proposition, we turn next to the consideration of the remaining fundamental conjunctive forms. These will be classed under the one head *Composite* for reasons which will be apparent later. So far, compound propositions have been divided into the two species *Conjunctive* and *Composite*, and we shall now proceed to subdivide the latter into four sub-species, each of which has its appropriate conjunctive expression, viz.:

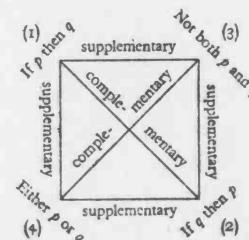
- Complementary (1) The Direct-Implicative function of  $p, q$ :— If  $p$  then  $q$ .  
 Complementary (2) The Counter-Implicative function of  $p, q$ :— If  $q$  then  $p$ .  
 (3) The Disjunctive function of  $p, q$ :— Not-both  $p$  and  $q$ .  
 (4) The Alternative function of  $p, q$ :— Either  $p$  or  $q$ .

In the implicative function 'If  $p$  then  $q$ ,'  $p$  is *implicans*<sup>1</sup> and  $q$  *implicate*; in the counter-implicative function 'If  $q$  then  $p$ ,'  $p$  is *implicate* and  $q$  is *implicans*<sup>1</sup>; in the dis-

<sup>1</sup> The plural of *implicans* must be written: *implicants*.

junctive function 'Not-both  $p$  and  $q$ ,'  $p$  and  $q$  are disjuncts; and in the alternative function 'Either  $p$  or  $q$ ,'  $p$  and  $q$  are alternants. These four functions of  $p, q$ , are distinct and independent of one another. The technical names that have been chosen are obviously in accordance with ordinary linguistic usage. The implicative and counter-implicative functions are said to be *Complementary* to one another, as also the disjunctive and alternative functions. Each of the four other pairs, viz. (1) and (3); (1) and (4); (2) and (3); (2) and (4) may be called a pair of *Supplementary* propositions. These names are conveniently retained in the memory by help of

### The Square of Independence



Now when we bring into antithesis the four conjunctive functions:

(1)  $p$  and not- $q$ ; (2) not- $p$  and  $q$ ; (3)  $p$  and  $q$ ; (4) not- $p$  and not- $q$ ;

with the four composite functions:

(1) if  $p$  then  $q$ ; (2) if  $q$  then  $p$ ; (3) not-both  $p$  and  $q$ ; (4) either  $p$  or  $q$ ;

we shall find that each of the composite propositions is *equivalent* to the negation of the corresponding conjunctive. This is directly seen in the case (3) of the conjunctive and the disjunctive functions of  $p, q$ . Thus,

*Complementary form*

*Composite form*



'Not-both  $p$  and  $q$ ' is the direct negative of 'Both  $p$  and  $q$ .' Again, for case (4) 'Either  $p$  or  $q$ ' is the obvious negative of 'Neither  $p$  nor  $q$ .' The relations of negation for *all* cases may be derived by first systematically tabulating the equivalences which hold amongst the composite functions, as below: abbreviating not- $p$  and not- $q$  into the forms  $\bar{p}$  and  $\bar{q}$  respectively.

*Table of Equivalences of the Composite Functions*

| Implicative<br>Form                              | Counter-<br>Implicative<br>Form | Disjunctive<br>Form                                  | Alternative<br>Form |
|--------------------------------------------------|---------------------------------|------------------------------------------------------|---------------------|
| 1. If $p$ then $q$ = If $\bar{q}$ then $\bar{p}$ |                                 | Not both $p$ and $q$ = Either $\bar{p}$ or $q$       |                     |
| 2. If $\bar{p}$ then $\bar{q}$ = If $q$ then $p$ |                                 | Not both $\bar{p}$ and $q$ = Either $p$ or $\bar{q}$ |                     |
| 3. If $p$ then $\bar{q}$ = If $q$ then $\bar{p}$ |                                 | Not both $p$ and $q$ = Either $\bar{p}$ or $\bar{q}$ |                     |
| 4. If $\bar{p}$ then $q$ = If $\bar{q}$ then $p$ |                                 | Not both $\bar{p}$ and $\bar{q}$ = Either $p$ or $q$ |                     |

In the above table it will be observed:

(a) That each composite function can be expressed in four equivalent forms: thus, any two propositions in the same row are equivalent, while any two propositions in different rows are distinct and independent.

(b) That the propositions represented along the principal diagonal are expressed in terms of the *positive* components  $p, q$ ; being in fact identical respectively with the implicative, the counter-implicative, the disjunctive and the alternative functions of  $p, q$ .

(c) That all the remaining propositions are expressed as functions of  $p$  and not- $q$ , or of not- $p$  and  $q$ , or of not- $p$  and not- $q$ .

We may translate the equivalences tabulated above in the form of equivalences of functions, thus:

1. The implicative function of  $p, q$ ; the counter-implicative function of not- $p$ , not- $q$ ; the disjunctive func-

tion of  $p$ , not- $q$ ; and the alternative function of not- $p, q$ , are all equivalent to one another. Again,

2. The counter-implicative function of  $p, q$ ; the implicative function of not- $p$ , not- $q$ ; the disjunctive function of not- $p, q$ ; and the alternative function of  $p$ , not- $q$ , are all equivalent to one another. Again,

3. The disjunctive function of  $p, q$ ; the alternative function of not- $p$ , not- $q$ ; the implicative function of  $p$ , not- $q$ ; and the counter-implicative function of not- $p, q$ , are all equivalent to one another. Again,

4. The alternative function of  $p, q$ ; the disjunctive function of not- $p$ , not- $q$ ; the implicative function of not- $p, q$ ; and the counter-implicative function of  $p$ , not- $q$ , are all equivalent to one another.

Since the force of each of the four composite functions of  $p, q$  can be represented by using either the Implicative or the Counter-implicative or the Disjunctive or the Alternative form, the classification of the four functions under one head *Composite* is justified. And since each Composite function is equivalent to a certain *Disjunctive* proposition, it is also equivalent to the negation of the corresponding *Conjunctive* proposition. Thus:

- |                            |                         |                         |                               |
|----------------------------|-------------------------|-------------------------|-------------------------------|
| 1. The implicative         | 'If $p$ then $q$ '      | negates the Conjunctive | ' $p$ and $\bar{q}$ '         |
| 2. The counter-implicative | 'If $q$ then $p$ '      | " "                     | " ' $\bar{p}$ and $q$ '       |
| 3. The disjunctive         | 'Not both $p$ and $q$ ' | " "                     | " ' $p$ and $q$ '             |
| 4. The alternative         | 'Either $p$ or $q$ '    | " "                     | " ' $\bar{p}$ and $\bar{q}$ ' |

Thus inasmuch as *no* composite function is *equivalent* to any conjunctive function, we have justified our division of compound propositions into the two fundamentally opposed species *Conjunctive* and *Composite*.



The distinction and relation between these composite forms of proposition may be further brought out by tabulating the inferences in which a simple conclusion is drawn from the conjunction of a composite with a simple premiss. In traditional logic, the Latin verbs *ponere* (to lay down or *assert*) and *tollere* (to raise up or *deny*) have been used in describing these different modes of argument. The gerund *ponendo* (by affirming) or *tollendo* (by denying) indicates the nature of the (simple) *premiss* that occurs; while the participle *ponens* (affirming) or *tollens* (denying) indicates the nature of the (simple) *conclusion* that occurs: the validity or invalidity of the argument depending on the nature of the *composite* premiss. There are therefore four modes to be considered corresponding to the four varieties of the composite proposition, thus:

Table of Valid Modes

| Modus                 |                                                        | Form of Composite Premiss |
|-----------------------|--------------------------------------------------------|---------------------------|
| 1. Ponendo Ponens :   | If $p$ then $q$ ; but $p$ ; $\therefore q$             | The Implicative.          |
| 2. Tollendo Tollens : | If $q$ then $p$ ; but $\bar{p}$ ; $\therefore \bar{q}$ | The Counter-Implic.       |
| 3. Ponendo Tollens :  | Not both $p$ and $q$ ; but $p$ ; $\therefore \bar{q}$  | The Disjunctive.          |
| 4. Tollendo Ponens :  | Either $p$ or $q$ ; but $\bar{p}$ ; $\therefore q$     | The Alternative.          |

The customary fallacies in inferences of this type may be exhibited as due to the confusion between a composite proposition and its complementary:

Table of Invalid Modes

| Modus                 |                                                        | Form of Composite Premiss |
|-----------------------|--------------------------------------------------------|---------------------------|
| 1. Ponendo Ponens :   | If $q$ then $p$ ; but $p$ ; $\therefore q$             | The Counter-Implic.       |
| 2. Tollendo Tollens : | If $p$ then $q$ ; but $\bar{p}$ ; $\therefore \bar{q}$ | The Implicative.          |
| 3. Ponendo Tollens :  | Either $p$ or $q$ ; but $p$ ; $\therefore \bar{q}$     | The Alternative.          |
| 4. Tollendo Ponens :  | Not both $p$ and $q$ ; but $\bar{p}$ ; $\therefore q$  | The Disjunctive.          |

The rules for correct inference from the above table of valid modes may be thus stated:

1. From an implicative, combined with the affirmation of its implicans, we may infer the affirmation of its implicate.

2. From an implicative, combined with the denial of its implicate, we may infer the denial of its implicans.

3. From a disjunctive, combined with the affirmation of one of its disjuncts, we may infer the denial of the other disjunct.

4. From an alternative, combined with the denial of one of its alternants, we may infer the affirmation of the other alternant.

§ 6. We ought here to refer to an historic controversy as regards the interpretation of the conjunction 'or.' It has been held by one party of logicians that what I have called the Alternative form of proposition, viz., that expressed by either-or, should be interpreted so as to include what I have called the Disjunctive, viz., that expressed by not-both. This view has undoubtedly been (perhaps unwittingly) fostered by the almost universal misemployment of the term *Disjunctive* to stand for what ought to be called Alternative. This prevalent confusion in terminology has led to a real blunder committed by logicians. The blunder consists in the fallacious use of the Ponendo Tollens as exhibited in the table above given. Consider the argument: ' $A$  will be either first or second'; 'It is found that  $A$  is second'; therefore ' $A$  is not first.' Here the conclusion is represented as following from the promising qualifications of the candidate  $A$ , whereas it really follows from the premiss ' $A$  cannot be both first and second.' In fact, the Alternative

proposition which is put as premiss is absolutely irrelevant to the conclusion, which would be equally correctly inferred whether the alternative predication were false or true.

It remains then to consider whether the logician can properly impose the one interpretation of the alternative form of proposition rather than the other. The reply here, as in other similar cases, is that, in the matter of verbal interpretation, the logician can impose legislation—not upon others—but only upon himself. However, where any form of verbal expression is admittedly ambiguous, it is better to adopt the interpretation which gives the smaller rather than the greater force to a form of proposition, since otherwise there is danger of attaching to the judgment an item of significance beyond that intended by the asserter. This principle of interpretation has the further advantage that it compels the speaker when necessary to state unmistakably and explicitly what may have been implicitly and perhaps confusedly present in his mind. I have therefore adopted as my interpretation of the form *Either-or* that smaller import according to which it does *not* include *Not-both*. Those logicians who have insisted on what is called the 'exclusive' interpretation of the alternative form of proposition (i.e. the interpretation according to which *Either-or* includes *Not-both*) seem sometimes to have been guilty of a confusion between what a proposition asserts, and what may happen to be known independently of the proposition. Thus it may very well be the case that the alternants in an alternative proposition are almost always 'exclusive' to one another; but this, so far from proving that the alternative proposition *affirms*

*Bradley's Example*

this exclusiveness, rather suggests that the exclusiveness is a fact commonly known independently of the special information supplied by the alternative proposition itself.

In this connection, the significance of the term complementary which I have applied to the implicative and counter-implicative as well as to the disjunctive and alternative, may be brought out. Propositions are appropriately called complementary when a special importance attaches to their conjoint assertion<sup>1</sup>. Thus it may be regarded as an ideal of science to establish a pair of propositions in which the implicans of the one is the implicate of the other; and again to establish a number of propositions which are mutually co-disjunct and collectively co-alternate. The term complementary is especially applicable where propositions are conjoined in either of these ways, because separately the propositions represent the fact partially, and taken together they represent the same fact with relative completeness.

We next consider the inferences that can be drawn from the conjunction of two *supplementary* propositions. These may be tabulated in two forms, the first of which brings out the fundamental notion of the *Dilemma*; and the second that of the *Reductio ad Impossibile*.

Dilemma:

*First Table for the Conjunction of Supplementaries*  
*The Dilemma*

- (1) 'If  $p$  then  $q$ ' and (4) 'If  $\bar{p}$  then  $q$ ': therefore,  $q$ .  
 (3) 'If  $p$  then  $\bar{q}$ ' and (2) 'If  $\bar{p}$  then  $\bar{q}$ ': therefore,  $\bar{q}$ .  
 (2) 'If  $q$  then  $p$ ' and (4) 'If  $\bar{q}$  then  $p$ ': therefore,  $p$ .  
 (3) 'If  $q$  then  $\bar{p}$ ' and (1) 'If  $\bar{q}$  then  $\bar{p}$ ': therefore,  $\bar{p}$ .

<sup>1</sup> Thus complementary propositions might be defined as those which are frequently confused in thought and frequently conjoined in fact.

The above table illustrates the following principle:

The conjunction of two implicatives, containing a common implicate but contradictory implicants, yields the affirmation of the simple proposition standing as common implicate. Or otherwise:

Any proposed proposition must be true when its truth would be implied both by the supposition of the truth and by the supposition of the falsity of some other proposition.

*Second Table for the Conjunction of Supplementaries  
The Reductio ad Impossibile*

- (1) 'If  $\bar{q}$  then  $\bar{p}$ ' and (4) 'If  $\bar{q}$  then  $p$ ': therefore,  $q$ .  
 (3) 'If  $q$  then  $\bar{p}$ ' and (2) 'If  $q$  then  $p$ ': therefore,  $\bar{q}$ .  
 (2) 'If  $\bar{p}$  then  $\bar{q}$ ' and (4) 'If  $\bar{p}$  then  $q$ ': therefore,  $p$ .  
 (3) 'If  $p$  then  $\bar{q}$ ' and (1) 'If  $p$  then  $q$ ': therefore,  $\bar{p}$ .

This second table illustrates the following principle:

The conjunction of two implicatives, containing a common implicans but contradictory implicates, yields the denial of the simple proposition standing as common implicans. Or otherwise:

Any proposed proposition must be false when the supposition of its truth would imply (by one line of argument) the truth and (by another line of argument) the falsity of some other proposition.

§ 7. In tabulating the formulae for Composite propositions as above I have merely systematised (with slight extensions and modifications of terminology) what has been long taught in traditional logic; and it is only in these later days that criticisms have been directed against the traditional formulae, especially on the ground that their uncritical acceptance has been found

to lead to certain paradoxical consequences, which may be called the Paradoxes of Implication. In this connection it is (I think) desirable to explain what is meant by a paradox. When a thinker accepts step by step the principles or formulae propounded by the logician until a formula is reached which conflicts with his common-sense, then it is that he is confronted with a paradox. The paradox arises—not from a merely blind submission to the authority of logic, or from any arbitrary or unusual use of terms on the logician's part—but from the very nature of the case, as apprehended in the exercise of powers of reasoning with which everyone is endowed. In particular, the paradoxes of implication are not due to any unnatural use of the *term* implication, nor to the positing of any *fundamental* formula that appears otherwise than acceptable to common sense. It is the formulae that are *derived*—by apparently unexceptionable means from apparently unexceptionable first principles—that appear to be exceptionable.

"Paradox."

Let us trace the steps by which we reach a typical paradox. Consider the alternative 'Not- $p$  or  $q$ .' If this alternative were conjoined with the assertion ' $p$ ,' we should infer ' $q$ .' Hence, 'Not- $p$  or  $q$ ' is equivalent to 'If  $p$  then  $q$ .' Similarly ' $p$  or  $q$ ' is equivalent to 'If not- $p$  then  $q$ .' Now it is obvious that the less determinate statement ' $p$  or  $q$ ' could always be inferred from the more determinate statement ' $p$ ': e.g. from the relatively determinate statement ' $A$  is a solicitor' we could infer ' $A$  is a solicitor or a barrister' i.e. ' $A$  is a lawyer.' Hence, whatever proposition  $q$  may stand for, we can infer ' $p$  or  $q$ ' from ' $p$ '; or again, whatever  $p$  may stand

for we can infer 'not- $p$  or  $q$ ' from  $q$ . Hence (i) given ' $p$ ' we may infer 'If not- $p$  then  $q$ ,' and (ii) given ' $q$ ' we may infer 'If  $p$  then  $q$ ,' whatever propositions  $p$  and  $q$  may stand for. These two consequences of the uncritical acceptance of traditional formulae have been expressed thus: (i) A *false* proposition (e.g. not- $p$  when  $p$  has been asserted) implies *any* proposition (e.g.  $q$ ); (ii) A *true* proposition (e.g.  $q$ , when  $q$  has been asserted) is implied by *any* proposition (e.g.  $p$ ). Thus ' $2 + 3 = 7$ ' would imply that 'It will rain to-morrow'; and 'It will rain to-morrow' would imply that ' $2 + 3 = 5$ .' That these two implicative statements are technically correct is shown by translating them into their equivalent alternative forms, viz.: (i) 'Either  $2 + 3$  is unequal to 7 or it will rain to-morrow'; (ii) 'Either it will not rain to-morrow or  $2 + 3 = 5$ .' We may certainly say that one or other of the two alternants in (i) as also in (ii) is true, the other being of course doubtful.

Taking 'If  $p$  then  $q$ ' to stand for the paradoxically reached implicative in both cases, we have shown that (i) from the denial of  $p$  (the implicans), and (ii) from the affirmation of  $q$  (the implicate) we may pass to the assertion 'If  $p$  then  $q$ .' This is, of course, only another way of saying that the implicative 'If  $p$  then  $q$ ' is equivalent to the alternative ' $p$  false or  $q$  true.' Thus when we know that 'If  $p$  then  $q$ ' is true, it follows that we know that 'either  $p$  is false or  $q$  is true'; but it does *not* follow that either 'we know that  $p$  is false' or 'we know that  $q$  is true.' The paradoxically reached implicative merely brings out the fact that this may be so *in some cases*: i.e. when asserting 'If  $p$  then  $q$ ,' there are cases in which we know that ' $p$  is false,' and there

are cases in which we know that ' $q$  is true.' But it is proper to enquire whether in actual language—literary or colloquial—the implicative form of proposition is ever introduced in this paradoxical manner. On the one hand, we find such expressions as: 'If that boy comes back, I'll eat my head'; 'If you jump over that hedge, I'll give you a thousand pounds'; 'If universal peace is to come tomorrow, the nature of mankind must be very different from what philosophers, scientists and historians have taken it to be'; etc., etc. Such phrases are always interpreted as expressing the *speaker's* intention to *deny the implicans*; the reason being that the *hearer* is assumed to be ready to deny the *implicate*. Again, on the other hand, we find such forms as: 'If Shakespeare knew no Greek, he was not incapable of creating great tragedies.' 'If Britain is a tiny island, on the British Empire the sun never sets.' 'If Boswell was a fool, he wrote a work that will live longer than that of many a wiser man.' 'If Lloyd George has had none of the advantages of a public school education, it cannot be maintained that he is an unintelligent politician.' Such phrases are always interpreted as expressing the *speaker's* intention to *affirm the implicate*, the reason being that the *hearer* may be assumed to be willing to *affirm the implicans*.

Looking more closely into the matter we find that when a speaker adopts the implicative form to express his denial of the implicans, he tacitly expects his hearer to supplement his statement with a *tollendo tollens*; and when he adopts it to express his affirmation of the implicate, he expects the hearer to supplement it with a *ponendo ponens*. Furthermore, inasmuch as the alternative form of proposition requires to be supplemented

by a *tollendo* (*ponens*) and the disjunctive by a *ponendo* (*tollens*), we find that an implicative intended to express the denial of its implicans is quite naturally expressed otherwise as an alternative: e.g. 'That boy won't come back or I'll eat my head,' to which the hearer is supposed to add 'But you won't eat your head'; therefore (I am to believe that) 'the boy won't come back' (*tollendo ponens*); and we find that an implicative intended to express the affirmation of its implicate is quite naturally expressed otherwise as a disjunctive: e.g. 'It cannot be held that Shakespeare both knew no Greek and was incapable of creating great tragedies,' to which the hearer is supposed to add 'But Shakespeare knew no Greek,' and therefore (I am to believe that) 'he was capable of creating great tragedies' (*ponendo tollens*).

We have yet to explain how the appearance of paradox is to be removed in the general case of a composite being inferred from the denial of an implicans (or disjunct) or from the affirmation of an implicate (or alternant). Now the ordinary purpose to which an implicative (or, more generally, a composite) proposition is put is *inference*: so much so that most persons would hesitate to assert the relation expressed in a composite proposition unless they were prepared to use it for purposes of inference in one or other of the four modes, *ponendo ponens*, etc. In other words, Implication is naturally regarded as tantamount to Potential Inference. Now when (i) we have inferred 'If  $p$  then  $q$ ' from the denial of ' $p$ ,' can we proceed from 'If  $p$  then  $q$ ' conjoined with ' $p$ ' to infer ' $q$ '? In this case we join the affirmation of ' $p$ ' with a premiss which has been inferred from the denial of  $p$ ; and this involves *Contra-*

*diction*, so that such an inference is impossible. Again, when (ii) we have inferred 'If  $p$  then  $q$ ' from the affirmation of ' $q$ ,' can we proceed from 'If  $p$  then  $q$ ' conjoined with ' $p$ ' to infer ' $q$ '? In this case we profess to infer ' $q$ ' by means of a premiss which was itself inferred from ' $q$ '; and this involves *Circularity*, so that this inference again must be rejected. The solution of the paradox is therefore found in the consideration that though we may correctly infer an implicative from the denial of its implicans, or from the affirmation of its implicate, or a disjunctive from the denial of one of its disjuncts, or an alternative from the affirmation of one of its alternants, yet the implicative, disjunctive or alternative so reached cannot be applied for purposes of further inference without committing the logical fallacy either of contradiction or of circularity. Now it must be observed that the rhetorical or colloquial introduction of a paradoxical composite, which is meant to be interpreted as the simple affirmation or denial of one of its components, achieves its intention by introducing—as the other component of the composite—a proposition whose falsity or truth (as the case may be) is palpably obvious to the hearer. The hearer is then expected to supplement the composite by joining it with the obvious affirmation or denial of the added component, and thereby, in interpreting the intention of the speaker, to arrive at the proposition as conclusion which the speaker took as his first premiss. Accordingly the process of interpretation consists in taking the same propositions in the same mode and arrangement as would have entailed circularity if adopted by the speaker.

§ 8. The distinction between an implicative pro-



position that can and one that cannot be used for inferential purposes may now be further elucidated by reference to the distinction between Hypothesis and Assertion. In order that an implicative may be used for inference, both the implicans and the implicate must be entertained hypothetically. In the case of *ponendo ponens* the process of inference consists in passing to the *assertion* of the implicate by means of the *assertion* of the implicans, so that the propositions that were entertained hypothetically in the implicative, come to be adopted assertively in the process of inference. The same holds, *mutatis mutandis*, for the other modes. Now when we have inferred an implicative from the affirmation of its implicate or from the denial of its implicans—as in the case of the implicative which appears paradoxical—the two components of the implicative thus reached cannot both be regarded as having been entertained hypothetically; and hence the principle according to which inference is a process of passing from propositions entertained hypothetically to the same propositions taken assertorically, would be violated if we used the composite for inference. This consideration constitutes a further explanation of how the paradoxes in question are solved.

The above analysis may be symbolically represented by placing under the letter standing for a proposition the sign  $\vdash$  to stand for assertorically adopted and the sign  $\text{H}$  for hypothetically entertained.

Thus the fundamental formula for correct inference may be rendered:

From ' $\text{H}$  would imply  $\text{H}$  with  $\text{H}$ ; we may infer  $\text{H}$ ,

where, in the implicative premiss, both implicans and implicate are entertained hypothetically.

Now the following inferences, which lead to paradoxical consequences, may be considered correct: i.e.

(a) From  $\text{H}$ , we may infer ' $\text{H}$  would imply  $\text{H}$ .'

(b) From  $\text{H}$ , we may infer ' $\text{H}$  would imply  $\text{H}$ .'

But the implicative conclusions here reached cannot be used for further inference: i.e.

(c) From ' $\text{H}$  would imply  $\text{H}$ ' with  $\text{H}$ ; we cannot infer  $\text{H}$ .

(d) From ' $\text{H}$  would imply  $\text{H}$ ' with  $\text{H}$ ; we cannot infer  $\text{H}$ .

For in (c) the implicate, and in (d) the implicans enters assertorically, and these inferences therefore contravene the above fundamental formula which requires that both implicate and implicans should enter hypothetically. Thus while admitting (a) that 'a true proposition would be implied by *any* proposition,' yet we cannot admit (c) that 'a true proposition *can be inferred* from any proposition.' Similarly, while admitting (b) that 'a false proposition would imply *any* proposition,' yet we cannot admit (d) that 'from a false proposition we can *infer* any proposition.' In fact, the attempted inference (c), where the conclusion has already been asserted, would entail *circularity*; and the attempted inference (d), where the premiss has already been denied, would involve *contradiction*.

Still maintaining the equivalence of the composite propositions expressible in the implicative, the counter-implicative, the alternative or the disjunctive form, each of these four forms will give rise to a like paradox. The



following table gives all the cases in which we reach a Paradoxical Composite; that is, a Composite which cannot be used for inference, either in the modus *ponendo ponens*, *tollendo tollens*, *ponendo tollens* or *tollendo ponens*. The sign of assertion in each composite must be interpreted to mean asserted to be true when the term to which it is attached *agrees* with the premiss, and asserted to be false when it *contradicts* the premiss.

Table of Paradoxical Composites

- (a) From  $q$  we may properly infer  
 $\vdash$   
 (1)  $\bar{p}$  or  $q = \text{If } \bar{p} \text{ then } q = \text{If } \bar{q} \text{ then } \bar{p} = \text{Not both } \bar{p} \text{ and } \bar{q},$   
 $\text{H} \quad \vdash \quad \text{H} \quad \vdash \quad \text{H} \quad \vdash \quad \text{H}$   
 or (2)  $\bar{p}$  or  $q = \text{If } \bar{p} \text{ then } q = \text{If } \bar{q} \text{ then } \bar{p} = \text{Not both } \bar{p} \text{ and } \bar{q}.$   
 $\text{H} \quad \vdash \quad \text{H} \quad \vdash \quad \text{H} \quad \vdash \quad \text{H}$
- (b) From  $\bar{q}$  we may properly infer  
 $\vdash$   
 (3)  $\bar{p}$  or  $\bar{q} = \text{If } \bar{p} \text{ then } \bar{q} = \text{If } q \text{ then } \bar{p} = \text{Not both } \bar{p} \text{ and } q,$   
 $\text{H} \quad \vdash \quad \text{H} \quad \vdash \quad \text{H} \quad \vdash \quad \text{H}$   
 or (4)  $\bar{p}$  or  $\bar{q} = \text{If } \bar{p} \text{ then } \bar{q} = \text{If } q \text{ then } \bar{p} = \text{Not both } \bar{p} \text{ and } \bar{q}.$   
 $\text{H} \quad \vdash \quad \text{H} \quad \vdash \quad \text{H} \quad \vdash \quad \text{H}$

The above composites can never be used for further inference. Thus:

in line (1), the attempted inference

' $\bar{p} \therefore q$ ' would be circular and 'not- $q \therefore$  not- $\bar{p}$ ' would be contradictory;

in line (2), the attempted inference

'not- $\bar{p} \therefore q$ ' would be circular, and 'not- $q \therefore \bar{p}$ ' would be contradictory;

in line (3), the attempted inference

'not- $\bar{p} \therefore$  not- $q$ ' would be circular, and ' $q \therefore \bar{p}$ ' would be contradictory;

in line (4), the attempted inference

' $\bar{p} \therefore$  not- $q$ ' would be circular, and ' $q \therefore$  not- $\bar{p}$ ' would be contradictory.

The paradox of implication assumes many forms, some of which are not easily recognised as involving mere varieties of the same fundamental principle. But

I believe that they can all be resolved by the consideration that we cannot *without qualification* apply a composite and (in particular) an implicative proposition to the further process of inference. Such application is possible only when the composite has been reached irrespectively of any *assertion* of the truth or falsity of its components. In other words, it is a necessary condition for further inference that the components of a composite should really have been entertained hypothetically when asserting that composite.

§ 9. The theory of compound propositions leads to a special development when in the conjunctives the components are taken—not, as hitherto, assertorically—but hypothetically as in the composites. The conjunctives will now be naturally expressed by such words as possible or compatible, while the composite forms which respectively contradict the conjunctives will be expressed by such words as necessary or impossible. If we select the negative form for these conjunctives, we should write as contradictory pairs:

| Conjunctives (possible)                 | Composites (necessary)              |
|-----------------------------------------|-------------------------------------|
| a. $\bar{p}$ does not imply $q$         | a. $\bar{p}$ implies $q$            |
| b. $\bar{p}$ is not implied by $q$      | b. $\bar{p}$ is implied by $q$      |
| c. $\bar{p}$ is not co-disjunct to $q$  | c. $\bar{p}$ is co-disjunct to $q$  |
| d. $\bar{p}$ is not co-alternate to $q$ | d. $\bar{p}$ is co-alternate to $q$ |

Or otherwise, using the term 'possible' throughout, the four conjunctives will assume the form that the several conjunctions— $\bar{p}\bar{q}$ ,  $\bar{p}q$ ,  $p\bar{q}$  and  $pq$ —are respectively *possible*. Here the word *possible* is equivalent to being merely hypothetically entertained, so that the several conjunctives are now qualified in the same way as are the simple components themselves. Similarly the four

corresponding composites may be expressed negatively by using the term 'impossible,' and will assume the form that the conjunctions  $p\bar{q}$ ,  $\bar{p}q$ ,  $pq$  and  $\bar{p}\bar{q}$  are respectively *impossible*, or (which means the same) that the disjunctions  $p\bar{q}$ ,  $\bar{p}q$ ,  $pq$  and  $\bar{p}\bar{q}$  are *necessary*. Now just as 'possible' here means merely 'hypothetically entertained,' so 'impossible' and 'necessary' mean respectively 'assertorically denied' and 'assertorically affirmed.'

The above scheme leads to the consideration of the determinate relations that could subsist of  $p$  to  $q$  when these eight propositions (conjunctives and composites) are combined in every possible way without contradiction. *Primâ facie* there are 16 such combinations obtained by selecting  $a$  or  $\bar{a}$ ,  $b$  or  $\bar{b}$ ,  $c$  or  $\bar{c}$ ,  $d$  or  $\bar{d}$  for one of the four constituent terms. Out of these 16 combinations, however, some will involve a conjunction of supplementaries (see tables on pp. 37, 38), which would entail the assertorical affirmation or denial of one of the components  $p$  or  $q$ , and consequently would not exhibit a *relation* of  $p$  to  $q$ . The combinations that, on this ground, must be disallowed are the following *nine*:

$\bar{a}\bar{b}\bar{c}\bar{d}$ ,  $a\bar{b}\bar{c}\bar{d}$ ,  $\bar{a}b\bar{c}\bar{d}$ ,  $\bar{a}b\bar{c}d$ ;  $\bar{a}b\bar{c}d$ ,  $\bar{a}b\bar{c}\bar{d}$ ,  $\bar{a}b\bar{c}d$ ,  $\bar{a}b\bar{c}d$ ;  $\bar{a}b\bar{c}d$ ,  $\bar{a}b\bar{c}d$ ,  $\bar{a}b\bar{c}d$ ,  $\bar{a}b\bar{c}d$ .

The combinations that remain to be admitted are therefore the following *seven*:

$a\bar{b}\bar{c}\bar{d}$ ,  $a\bar{b}\bar{c}d$ ;  $a\bar{b}\bar{c}d$ ,  $a\bar{b}\bar{c}d$ ;  $a\bar{b}\bar{c}d$ ,  $a\bar{b}\bar{c}d$ ;  $a\bar{b}\bar{c}d$ ,  $a\bar{b}\bar{c}d$ ;  $a\bar{b}\bar{c}d$ ,  $a\bar{b}\bar{c}d$ ;  $a\bar{b}\bar{c}d$ ,  $a\bar{b}\bar{c}d$ .

In fact, under the imposed restriction, since  $a$  or  $b$  cannot be conjoined with  $c$  or  $d$ , it follows that we must always conjoin  $a$  with  $\bar{c}$  and  $\bar{d}$ ;  $b$  with  $\bar{c}$  and  $\bar{d}$ ;  $c$  with  $\bar{a}$  and  $\bar{b}$ ;  $d$  with  $\bar{a}$  and  $\bar{b}$ . This being understood, the

seven permissible combinations that remain are properly to be expressed in the more simple forms:

$ab$ ,  $cd$ ;  $a\bar{b}$ ,  $b\bar{a}$ ,  $c\bar{d}$ ,  $d\bar{c}$ ; and  $\bar{a}\bar{b}\bar{c}\bar{d}$ .

These will be represented (but re-arranged for purposes of symmetry) in the following table giving all the possible relations of any proposition  $p$  to any proposition  $q$ . The technical names which I propose to adopt for the several relations are printed in the second column of the table.

*Table of possible relations of proposition p to proposition q.*

|                                                                                                                              |                                               |
|------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------|
| 1. $(a, b)$ : $p$ implies and is implied by $q$ .                                                                            | $p$ is co-implicant to $q$ . (equipollent)    |
| 2. $(a, \bar{b})$ : $p$ implies but is not implied by $q$ .                                                                  | $p$ is super-implicant to $q$ . (superaltern) |
| 3. $(\bar{b}, a)$ : $p$ is implied by but does not imply $q$ .                                                               | $p$ is sub-implicant to $q$ . (subaltern)     |
| 4. $(\bar{a}, \bar{b}, \bar{c}, \bar{d})$ : $p$ is neither implicant nor implicate nor co-disjunct nor co-alternate to $q$ . | $p$ is independent of $q$ . (independent)     |
| 5. $(d, \bar{c})$ : $p$ is co-alternate but not co-disjunct to $q$ .                                                         | $p$ is sub-opponent to $q$ . (sub-contrary)   |
| 6. $(c, \bar{d})$ : $p$ is co-disjunct but not co-alternate to $q$ .                                                         | $p$ is super-opponent to $q$ . (contrary)     |
| 7. $(c, d)$ : $p$ is co-disjunct and co-alternate to $q$ .                                                                   | $p$ is co-opponent to $q$ . (contradictory)   |

Here the symmetry indicated by the prefixes, co-, super-, sub-, is brought out by reading downwards and upwards to the middle line representing independence. In this order the propositional forms range from the supreme degree of consistency to the supreme degree of opponency, as regards the relation of  $p$  to  $q$ . In traditional logic the seven forms of relation are known respectively by the names equipollent, superaltern, subaltern, independent, sub-contrary, contrary, contradictory. This latter terminology, however, is properly used to express the *formal* relations of implication and opposition, whereas the terminology which I have adopted will apply indifferently both for formal and for *material* relations.

corresponding composites may be expressed negatively by using the term 'impossible,' and will assume the form that the conjunctions  $p\bar{q}$ ,  $\bar{p}q$ ,  $pq$  and  $\bar{p}\bar{q}$  are respectively *impossible*, or (which means the same) that the disjunctions  $p\bar{q}$ ,  $\bar{p}q$ ,  $pq$  and  $\bar{p}\bar{q}$  are *necessary*. Now just as 'possible' here means merely 'hypothetically entertained,' so 'impossible' and 'necessary' mean respectively 'assertorically denied' and 'assertorically affirmed.'

The above scheme leads to the consideration of the determinate relations that could subsist of  $p$  to  $q$  when these eight propositions (conjunctives and composites) are combined in every possible way without contradiction. *Primâ facie* there are 16 such combinations obtained by selecting  $a$  or  $\bar{a}$ ,  $b$  or  $\bar{b}$ ,  $c$  or  $\bar{c}$ ,  $d$  or  $\bar{d}$  for one of the four constituent terms. Out of these 16 combinations, however, some will involve a conjunction of supplementaries (see tables on pp. 37, 38), which would entail the assertorical affirmation or denial of one of the components  $p$  or  $q$ , and consequently would not exhibit a *relation* of  $p$  to  $q$ . The combinations that, on this ground, must be disallowed are the following *nine*:

$ab\bar{c}\bar{d}$ ,  $a\bar{b}\bar{c}d$ ,  $\bar{a}b\bar{c}d$ ,  $\bar{a}\bar{b}c\bar{d}$ ;  $\bar{a}b\bar{c}d$ ,  $\bar{b}a\bar{c}d$ ,  $\bar{c}a\bar{b}d$ ,  $\bar{d}a\bar{b}c$ ;  $ab\bar{c}d$ .

The combinations that remain to be admitted are therefore the following *seven*:

$ab\bar{c}\bar{d}$ ,  $cd\bar{a}\bar{b}$ ;  $a\bar{b}\bar{c}\bar{d}$ ,  $b\bar{a}\bar{c}\bar{d}$ ,  $c\bar{d}\bar{a}\bar{b}$ ,  $d\bar{c}\bar{a}\bar{b}$ ;  $\bar{a}\bar{b}\bar{c}d$ .

In fact, under the imposed restriction, since  $a$  or  $b$  cannot be conjoined with  $c$  or  $d$ , it follows that we must always conjoin  $a$  with  $\bar{c}$  and  $\bar{d}$ ;  $b$  with  $\bar{c}$  and  $\bar{d}$ ;  $c$  with  $\bar{a}$  and  $\bar{b}$ ;  $d$  with  $\bar{a}$  and  $\bar{b}$ . This being understood, the

seven permissible combinations that remain are properly to be expressed in the more simple forms:

$ab$ ,  $cd$ ;  $a\bar{b}$ ,  $b\bar{a}$ ,  $c\bar{d}$ ,  $d\bar{c}$ ; and  $\bar{a}\bar{b}\bar{c}\bar{d}$ .

These will be represented (but re-arranged for purposes of symmetry) in the following table giving all the possible relations of any proposition  $p$  to any proposition  $q$ . The technical names which I propose to adopt for the several relations are printed in the second column of the table.

*Table of possible relations of proposition p to proposition q.*

|                                                                                                                              |                                                        |
|------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------|
| 1. $(a, b)$ : $p$ implies and is implied by $q$ .                                                                            | $p$ is co-implicant to $q$ . ( <i>equipollent</i> )    |
| 2. $(a, \bar{b})$ : $p$ implies but is not implied by $q$ .                                                                  | $p$ is super-implicant to $q$ . ( <i>superaltern</i> ) |
| 3. $(\bar{b}, a)$ : $p$ is implied by but does not imply $q$ .                                                               | $p$ is sub-implicant to $q$ . ( <i>subaltern</i> )     |
| 4. $(\bar{a}, \bar{b}, \bar{c}, \bar{d})$ : $p$ is neither implicant nor implicate nor co-disjunct nor co-alternate to $q$ . | $p$ is independent of $q$ . ( <i>independent</i> )     |
| 5. $(d, \bar{c})$ : $p$ is co-alternate but not co-disjunct to $q$ .                                                         | $p$ is sub-opponent to $q$ . ( <i>sub-contrary</i> )   |
| 6. $(c, \bar{d})$ : $p$ is co-disjunct but not co-alternate to $q$ .                                                         | $p$ is super-opponent to $q$ . ( <i>contrary</i> )     |
| 7. $(c, d)$ : $p$ is co-disjunct and co-alternate to $q$ .                                                                   | $p$ is co-opponent to $q$ . ( <i>contradictory</i> )   |

Here the symmetry indicated by the prefixes, co-, super-, sub-, is brought out by reading downwards and upwards to the middle line representing independence. In this order the propositional forms range from the supreme degree of consistency to the supreme degree of opponency, as regards the relation of  $p$  to  $q$ . In traditional logic the seven forms of relation are known respectively by the names equipollent, superaltern, subaltern, independent, sub-contrary, contrary, contradictory. This latter terminology, however, is properly used to express the *formal* relations of implication and opposition, whereas the terminology which I have adopted will apply indifferently both for formal and for *material* relations.

## CHAPTER IV

## SECONDARY PROPOSITIONS AND MODALITY

§ 1. THE division of propositions into simple and compound is to be distinguished from another division to which we shall now turn, namely that into primary and secondary. A secondary proposition is one which predicates some characteristic of a primary proposition. While it is unnecessary to give a separate definition of a primary proposition, a tertiary proposition may be defined as one which predicates a certain characteristic of a secondary proposition, just as a secondary proposition predicates some characteristic of a primary proposition. Theoretically this succession of propositions of higher and higher order could be carried on indefinitely. But it should be observed that any adjective that can be predicated of a *primary* proposition can be significantly predicated of a proposition *as such*, i.e. equally of a primary, a secondary, and a tertiary, etc. proposition; and that, in consequence, although *propositions* may be ranged into higher and higher orders, adjectives predicable of propositions are of only one order, and will be called "pre-propositional." Taking  $p$  to stand for any proposition we may construct such secondary propositions as:  $p$  is true,  $p$  is false,  $p$  is certainly true,  $p$  is experientially certified,  $p$  has been maintained by Berkeley. Here we are predicating various adjectives (the precise meaning of which will be considered later)

of any given proposition  $p$ ; and we define each of these propositions—of which the subject-term is a proposition and the predicate-term an appropriate adjective—as secondary. One example may be given which has historic interest. Take (*A*) the proposition 'two straight lines cannot enclose a space'—to illustrate a primary proposition; again take (*B*) the proposition '*A* is established by experience' as a secondary proposition; and thirdly take (*C*) '*B* is held by Mill' as a tertiary proposition; namely—'It is held by Mill that the theorem that two straight lines cannot enclose a space is established solely by experience.' It is at once obvious that, all these three propositions, the primary, the secondary, the tertiary, which include the same matter (viz. that expressed in the primary) might be attacked or defended on totally distinct grounds. We may defend the primary proposition: 'two straight lines cannot enclose a space,' by showing perhaps that it is involved in the definition of 'straight'; again we might attack or defend the secondary proposition: 'this geometrical theorem is established by experience,' by considering the general nature of experience, and the possibilities of proving generalisations; and lastly, if we are to examine the tertiary proposition, namely 'Mill held the experiential view on the subject of this geometrical axiom,' we have only to read Mill's book and try, if possible, to understand what was the precise view that he wished to maintain.

§ 2. In connection with a larger and wider treatment of secondary propositions in general, it will be useful here to introduce the subject of Modality. We shall throughout speak of modal *adjectives*, instead of

modal *propositions*; it being understood that these adjectives fall under the general head of what we have called *pre-propositional* adjectives. We propose provisionally to include under modals the adjectives 'true' and 'false.' But a question of some interest arises as to whether the two very elementary cases '*p* is true' and '*p* is false' where *p* is a proposition are legitimate illustrations of secondary propositions. It may be held that the proposition '*p* is true' is in general reducible to the simple proposition *p*; so that, if this were so, '*p* is true' would only have the semblance of a secondary proposition, and would be equivalent for all ordinary purposes to the primary proposition *p*. It appears to me futile to enter into much controversy on this point, because it will be universally agreed that anyone who asserts the proposition *p* is implicitly committing himself to the assertion that *p* is true. And again the *consideration* of the proposition *p* is indistinguishable from the consideration of the proposition *p* as being true; or the attitude of *doubt* in regard to the proposition *p* simply means the attitude of doubt as regards *p* being true. These illustrations, in my view, show that we may say strictly that the adjective *true* is redundant as applied to the proposition *p*; which illustrates the principle, which I have put forward, that a proposition by itself is, in a certain sense, incomplete and requires to be supplemented by reference to the assertive attitude. Thus the assertion of *p* is equivalent to the assertion that *p* is true; though of course the *assertum* *p* is not the same as the *assertion* that *p* is true. The adjective *true* has thus an obvious analogy to the multiplier *one* in arithmetic: a number is unaltered when

multiplied by unity, and therefore in multiplication the factor *one* may be dropped; and in the same way the introduction of the adjective *true* may be dropped without altering the value or significance of the proposition taken as asserted or considered.

More interest attaches to the apparently secondary proposition '*p* is false.' It certainly appears that *p*-false is indistinguishable from not-*p*, and the majority of logicians rather assume that not-*p* is on a level with *p*, and may be at once co-ordinated with *p* as a primary proposition. Now it appears to me that, while *p*-true is practically indistinguishable from the primary proposition *p*, on the other hand *p*-false is essentially a secondary proposition, and can only be co-ordinated with primary propositions after a certain change of attitude has been adopted. This problem will come up again in the general treatment of negation and obversion.

§ 3. We may now turn to what have been always known as modal adjectives such as necessary, contingent, possible, etc. The discussion of modality is complicated rather unfortunately owing to certain merely formal confusions which have not been explicitly recognised. Hence, before plunging into the really difficult philosophic problems, these formal confusions must be cleared away. The simplest of these occurs in the controversy between those who hold that contradictories belong to the same sphere of modality, and those who hold that they belong to opposite spheres of modality. This controversy is resolved by explicitly realising the distinction between a primary and a secondary proposition. Thus taking, for purposes of illustration, the antithesis between necessary and con-

tingent, we may consider the primary proposition 'It is raining now' and its contradictory 'It is not raining now'; if one of these primary propositions is contingent, so also is the other. But the contradictory of the secondary proposition affirming *contingency* of the primary—i.e. 'that it is raining now is contingent'—is the secondary proposition which affirms *necessity* of the primary—i.e. 'that it is raining now is necessary.' Thus, in doubting or contradicting a *secondary* proposition, we use the opposite or contrary modal predicate; but in denying the *primary* proposition we should attach the same modal adjective to the proposition and to its contradictory. There can really be no difference of opinion on this subject; the *opposition* of modality is expressed in the *secondary* propositions that contradict one another; the *agreement* in modality holds of the *primary* propositions that contradict one another. Summarising: if a given primary proposition is necessarily true, its contradictory, which is also a primary proposition, is necessarily false; and if a given primary proposition is contingently true, its contradictory, which is also a primary proposition, is contingently false. Thus in both cases the contradictory primary propositions belong to the *same* sphere of modality. But the contradictory of a secondary proposition affirming necessity or contingency of a primary, will be the secondary proposition which affirms contingency or necessity of the primary. Thus the contradictories of the secondary propositions assert *opposite* modals.

It is necessary to enter into the more philosophical aspect of modality, if only in a preliminary and introductory way, because, apart from the confusion between

a secondary and a primary proposition, there is, it would appear, considerable confusion in regard to the terminology adopted by different logicians or philosophers in their treatment of modals. To do this we feel bound to reconsider entirely the terminology. Since Kant it has been customary to make a three-fold division, using the terms apodictic, assertoric, and problematic; and this trichotomous division at once leads to some unfortunate confusions. The precise significance of assertoric in particular is peculiarly ambiguous: thus the proposition '2 and 3 make 5' as it stands, would appear to be merely assertoric; so that assertoric would include apodictic as one of its species<sup>1</sup>. Let us then begin our investigation without any bias derived from the traditional terminology.

§ 4. The first antithesis that immediately impresses us in this connection is that between a *certified* and an *uncertified* proposition. A proposition which is *uncertified* appears to be what Kant and others have sometimes meant by a problematic proposition; hence we begin by replacing the term 'problematic' by the term 'uncertified.' The contradictory of uncertified is certified, so that all propositions may be divided into the two exclusive classes of certified and uncertified. It is of course obvious that these terms are what is called relative; that is to say, at one stage in the acquisition of knowledge a given proposition may be uncertified, while at a later or higher stage, or with increased opportunity of observation, etc., it may become certified. The

<sup>1</sup> This confusion is, of course, due merely to the failure to distinguish between a primary proposition as such and a secondary. It is totally independent of any question as to what the adjectives *assertoric* and *apodictic* mean respectively.



distinction therefore is of course not permanent or absolute, but temporal and relative to individuals and their means of acquiring knowledge. It might be held that such distinctions should be excluded from Logic; but this, in our opinion, is unsound, in as much as reference to the mental powers and the individual opportunities of acquiring knowledge turns out in many discussions to be a most essential topic for logical treatment. The whole doctrine of probability hinges upon our realising the changeable or relative opportunities and means, which differ, from one situation to another, in the extent of attainable knowledge. The further discussion then of uncertified propositions will later introduce the logical topic of probability. Returning to certified propositions, a distinction is required according as the given proposition is certified as true or certified as false; and thus we have a triple division: uncertified, certified as true, and certified as false. But for most purposes this latter distinction is unnecessary, because for the given proposition that has been certified as false we might substitute the contradictory proposition that has been certified as true. It would be enough therefore to use the two divisions uncertified and certified, understanding by certified 'certified as true.'

§ 5. The above division leads to a fundamentally important subdivision under the term 'certified'; for we must recognise, in epistemology or general philosophy, that there are essentially different principles or modes by which the truth of a proposition may be certified; and a rough two-fold classification will conveniently introduce this subject: thus we may contrast a proposition whose truth is certified by pure thought or reason

with a proposition which is certified on the ground of actual experience. Briefly we shall call these two classes 'formally certified' and 'experientially certified.' The range of these two modes of certification will be a matter of dispute: some philosophers hold that all the principles and formulae of logic, and all those of arithmetic and mathematics, are to be regarded as certified by pure thought or reason. This gives perhaps the widest range for the propositions that may be said to be formally certified. But even amongst these, we may have to distinguish those which *have* been formally certified, from amongst the entire range which may be regarded as formally *certifiable*. Others would hold that many mathematical principles, such as those of geometry, can only be certified by an appeal to sense-perception—a form of experience; and thus the limits to be ascribed to the range of formal certification would open up serious controversy. Again, on the other hand, the range of propositions immediately certifiable *by experience* raises serious problems. Some may hold that the only truths guaranteed by mere experience are the characterisations of actual sense-impressions experienced by the thinker at the moment in which he asserts the proposition; many would extend this to judgments on the individual's past experiences revived in memory; but the most universally understood range of experientially certified propositions is still wider: it would include sense-perceptions, and observations of physical phenomena, and even judgments on mental phenomena,—these supplying the required data for science in general. We will not then profess to draw the line precisely between propositions that are to be regarded as formally certifiable and those

that are to be regarded as experientially certifiable; but there is one explanation of the relation between these two classes which will probably be admitted by all; namely, that propositions which are admittedly based on experience, will also involve processes of thought or reasoning, and that therefore no propositions of any importance are based upon experience *alone*; since an element of thought or reason enters into the certification of all such propositions. This leads to a simple, more precise definition of the antithesis—formal and experiential: while we define a formally certifiable proposition as one which can be certified by thought or reason alone, we do not define experiential propositions as those which can be certified by experience *alone*, but rather as those which can *only* be certified with the aid of experience. In this way we imply that experience alone would be inadequate<sup>1</sup>.

A certain relation between the two antithetical modals, formal and experiential, will be found to apply over and over again to other antitheses in the characteristics of propositions. It may be illustrated by reference to the syllogism. Thus a certain syllogism may contain one formal premiss and one experiential premiss; and the conclusion deducible from these two premisses must be called experiential, because it has been certified by at least one experiential premiss. To put it otherwise, if all the premisses of an inference were formal,

<sup>1</sup> Even this distinction requires amendment; for it may be maintained that just as experience alone can certify nothing, so thought alone can certify nothing. Thus formal certification would coincide with what requires only *experience in general* (to use Kantian terminology) whilst experiential certification would involve in addition special or particular experience.

the conclusion would be formal; but if *only one* premiss is experiential (even though the others may be formal), the conclusion must be experiential. This particular characteristic of the syllogism is not arbitrary, but follows from the common understanding of what is meant by 'experientially certified,' namely something which could not be certified without experience,—*not* something which could be certified by experience *alone*.

§ 6. One of the chief sources of confusion is the use of the term 'necessary' in various different senses as an adjective predicable of propositions. It has sometimes been said that *all* propositions should be conceived as necessary; in the sense that the asserter of a proposition represents to himself an objective ground or reference to which he submits and which restrains the free exercise of his will in the act of judgment. This contention is indisputable, and may be regarded as one of the many ways in which the nature of judgment or assertion as such may be philosophically expounded. But obviously necessity as so conceived cannot serve as a predicate for distinguishing between propositions of different kinds. We pass, therefore, to the next and more usual meaning of the term necessary which will perhaps best be indicated by a quotation from Kant: 'Mathematical propositions are always judgments *a priori* and not empirical, because they carry with them the conception of necessity, which cannot be given by experience.' Here necessary is opposed to empirical; and the antithesis that Kant has in view coincides approximately with that between the *formally* certified and the *experientially* certified (as I have preferred to express it). But still another meaning has

been attached to the term necessary, viz., that according to which the necessary is opposed to the contingent. If, however, the term contingent is interpreted as equivalent to (what I have called) experientially certified, then we might agree that necessary should be interpreted as equivalent to formally certified; and in this case we should not have found a third meaning to the term. The question therefore arises whether a use can be found for the antithesis 'necessary' and 'contingent,' *within the sphere of the experientially certified*. Now it has been maintained as a fundamental philosophical postulate that 'All that happens is necessitated'; and this may be taken as equivalent to saying that 'Nothing that happens is contingent.' It should here be pointed out that this contention is to be clearly distinguished from the view that 'All judgments or propositions are necessary.' For the necessity ascribed to judgments is conceived as a compulsion exercised by the objective or real upon the thinker; whereas the necessitation attributed to events is conceived (more or less metaphorically) as a compulsion exercised by nature as a unity upon natural phenomena as a plurality. The former necessity is so to speak objective-subjective; the latter objective-objective. But an elementary criticism must be directed against the use made of the postulate 'All that happens is necessitated' to deduce that there is no proper scope for the term contingent. For we inevitably conceive of that which happens as being necessitated *by something else that happens* in accordance with (what is popularly called) a law of nature. In other words, the laws of nature *taken alone* do not necessitate any event whatever; we should have rather to say that a law of

nature necessitates that the happening of some one thing should necessitate the happening of a certain other thing. Hence, I should propose that *nomic* (from νόμος, a law) should be substituted for *necessary* as contrasted with contingent. Thus a nomic proposition is one that expresses a pure law of nature; and a contingent proposition is one that expresses a concrete event. In this way we have eliminated the ambiguous term necessary, and have substituted *formally certified* when the term is opposed to experientially certified; and *nomic* when opposed to contingent. Finally the term *possible* must be coupled with the word *necessary* in its three usages. For 'possible' has three obviously distinct meanings: (1) what is not known to be false; i.e. what does not contradict the necessary in the first sense, applicable to *all* assertions; (2) what does not conflict with any formally certified proposition, i.e. with any proposition necessary in the second sense; (3) what does not conflict with any law of nature, i.e. with any proposition necessary in the third sense. The word 'possible' in these three senses may be distinguished respectively as 'the *epistemically* possible,' 'the *formally* possible' and 'the *nomically* possible.'

§ 7. It will now be apparent that the antithesis between nomic and contingent is of a totally different nature from that between certified and uncertified, or between the different modes of certification. The latter has been called subjective, the former objective; but the terms epistemic and constitutive are preferable: for the characteristics 'nomic' and 'contingent' apply within the content of the proposition, and are therefore properly to be regarded as constitutive; whereas the character-

istics 'certified' and 'uncertified' apply to the relation of the proposition to the thinker, and should therefore be called epistemic. Taking for example, the proposition 'Nature is uniform': if this is held to be necessary in the sense that our reason alone establishes its truth, then the attribution of necessity is in this case of the same kind as what we have called formally certified and is thus epistemic. But the necessity involved in the laws of nature is generally attributed to Nature itself, and not merely to our grounds for asserting such uniformity: and is thus constitutive. Thus, if we say, as a specific example of the necessity attributed to Nature's processes, that 'bodies attract one another in obedience to the necessities of nature,' this statement is quite independent of any view we may hold as to the reasonable grounds for asserting the fact of universal gravitation. In short, referring back to the distinction between the fact and the proposition, such modals as certified and uncertified are adjectives directly characterising the *proposition*, whereas modals of the other kind, typified by nomic and contingent, directly characterise the *fact*.

§ 8. It remains now to introduce a certain familiar distinction amongst propositions not included in the understood meaning of modal, viz. that between real and verbal. These terms were used by Mill, and are generally understood as equivalent to Kant's terms 'synthetic' and 'analytic.' Mill's point of view is very different from Kant's, for Mill is thinking of the nature of language, of the definition of words, etc., while Kant is thinking of ideas and the various constructive acts of thought. Mill's usage is more easy to expound than

Kant's, and gives rise to less serious conflict of view. A verbal proposition is one which can be affirmed from a mere knowledge of the meanings of words and their modes of combination; a real proposition, on the other hand, requires for its acceptance, not only a knowledge of the meanings of words, but also a knowledge of matters of fact. We may therefore note the same relation between verbal and real as between formal and experiential<sup>1</sup>: namely, that two premisses, both of which are verbal, can only yield a verbal conclusion; and that a single real premiss, even though joined with any number of verbal premisses, will impose upon the conclusion its own character as real.

The definition so far given of verbal propositions seems fairly clear; it is therefore surprising that it should have proved a stumbling-block to some logicians. The people who have raised difficulty on this point are those who have preferred the Kantian terms 'analytic' and 'synthetic' in place of Mill's terms 'verbal' and 'real': ('analytic' Kant illustrates by the proposition 'Material bodies are extended,' 'synthetic' by the proposition 'Material bodies attract one another'). The controversy has arisen through a tacit confusion between 'verbal or analytic' and 'familiar' on the one hand, and between 'real or synthetic' and 'unfamiliar' on the other hand, due to the kind of examples chosen to illustrate each type of proposition. This confusion is apparent in the well-known dictum of Bradley—'that synthetic judgments are analytic in the making'—where it is clear that by a 'synthetic judgment' he means the newly-constructed proposition, and by 'in the making,'

<sup>1</sup> See above, last paragraph of § 5.

the process of rendering the proposition familiar. But, it needs only a little reflection to show that familiarity with a matter of fact does not render the proposition which expresses such fact verbal or analytic; nor does unfamiliarity with the meanings of words render a proposition which explains such meaning real or synthetic: a proposition about the meanings of words is verbal, and a proposition about matters of fact is real, whether the hearer is unfamiliar or familiar with the words or with the facts<sup>1</sup>. Thus the proposition '7 and 5 make 12' is familiar enough, but whether or not it is verbal (or analytic) has absolutely nothing to do with its familiarity; on the other hand, a technical definition given by a scientist will probably be quite unfamiliar, but if the scientist puts it forward as an expression of his intention to use a word with a certain significance, the proposition which states his intention is verbal, although it is *ipso facto* unfamiliar.

Perhaps a better way of indicating the nature of a verbal proposition, is to say that it is not quite what is ordinarily meant by a proposition; that is, as verbal, it cannot strictly be said to be either true or false, because it does not declare a fact, but rather expresses an intention, a command, or a request. The technical scientist puts forward his definitions in this spirit, when he asks readers to allow him to use a term with a cer-

<sup>1</sup> An important explanation of all this should be given. What Bradley means by "an analytic judgment"—not "a verbal proposition"—is a judgment that could be discovered by introspective analysis, so that his pronouncement is an obvious truism. But it is strange that he does not perceive that this is not in the least the same as what Kant meant. Kant's distinction is epistemological, Bradley's merely psychological.

tain signification which is explained by his definition. Thus a verbal proposition is neither true nor false, because it is properly expressed, not in the indicative, but in the imperative or other similar mood. But if by a verbal proposition is meant one that assigns the meaning of a word as conventionally used in any wider or narrower context, then, inasmuch as the proposition asserts the fact that such or such *is* the convention, it must be either true or false.

§ 9. At the beginning of this chapter we defined a secondary proposition as one that predicates one or other of the adjectives significantly predicated of a proposition as such. We proceeded to consider in turn different kinds of adjectives that are thus predicable: this has led to a discussion of modal adjectives, and has included in particular a consideration of the adjectives true and false, and finally of the predicates 'verbal' or 'analytic' and 'real' or 'synthetic.'



## CHAPTER V

## NEGATION

§ 1. UNDER the general problem of the nature of negation we may begin by considering the particular form of negation which has been called 'pure negation.' There appear to be several different meanings attached to the notion of *pure* negation: it may mean the simple attitude of rejection, as opposed to that of acceptance, towards a proposition taken as a unit and without further analysis. Such negation may be called pure, because the negative element does not enter within the content of the assertum, but expresses merely a certain mental attitude to the proposition itself. According to this definition of pure negation, the judgment which may be called purely negative has as its object precisely what I have called a secondary proposition in my previous discussion as to whether the statement '*p* is false' is to be regarded as primary or as secondary. When then we enquire as to the importance or the relevance of pure negation, we may be raising the question whether a judgment expressed in this purely negative form really ever represents a genuine attitude of thought. No doubt there are not many cases in which this negative attitude towards an assertum taken as a unit could be illustrated; but we may at least insist that, when some assertum is proposed which can be clearly conceived in thought, and yet repels any attempt to accept it, then the attitude

towards such an assertum to which our thinking process has led us is strictly to be called that of pure negation. For example, the proposition 'Matter exists' may appear to some philosophers to have in it a sufficiently clear content to enable them to reject it, without their having in mind any correspondingly clear substitute which they can accept. In this case their mental attitude towards the proposed assertum may be properly called one of mere negation; since the only positive element involved is the conceived content of the proposition rejected.

But the term pure negation is more generally applied where a predicate is denied of some subject *within* the proposition. Under this head, the case where negation would seem to be quite pure may be illustrated by a proposition like—'Wisdom is not blue.' Such a proposition would have purpose only in a logical context where we are pointing out that certain types of adjective cannot be predicated of certain types of substantive. A more common case which leads to a purely negative form of predication, is where, for instance, a distant object of perception, is considered as to whether it is blue or of some other colour, or as to whether it is a man or some other material body. Towards this proposed assertum—that it is blue, or that it is a man—our attitude may be that of mere denial, in the sense that we are perfectly clear what it is *not*, but we are not correspondingly clear as to what it is. We may admit that a judgment which in this sense is merely negative and without any positive content is rare, since when we deny of a flower that it is red, we are at least judging that it has some colour, and similarly when we deny of something in sight that it has the shape of a man we are



at least judging that it has some shape, and this constitutes a positive element in our judgment. The above examples illustrate two applications of the notion of negation: first, in denying the proposition as a whole, and again in denying that an adjective of a certain type can be predicated of a certain type of substantive, where the positive element is evanescent; and secondly, in denying the more specific predicate proposed for a substantive while tacitly asserting some wider predicate under which it falls, where a positive element is properly to be recognised.

Some logicians, going one step further, have asserted that, in denying an object to be red, not only is the *generic* adjective colour a positive factor in the judgment, but that some specific colour other than red is tacitly affirmed: that is, they hold that we cannot deny unless we have some positive *determinate* ground for our denial. But this reason for asserting the universal presence of a *positive* factor in judgment must not be confused with the former; for it is one thing to say that the denying of any proposed adjective involves the affirming of *some* other adjective of the same generic kind, and another thing to say that it involves the affirming of a *specific* adjective. While admitting the first, I reject the view that in denying red we are affirming say green or blue as the case may be, on the ground that it involves a confusion between what is necessarily determined in *fact* with what may or may not be determinate in our *knowledge of fact*. There are countless cases of our denying a certain proposed adjective in which, while we know that *some* determinate adjective can be truly applied, yet we do not know

*which* determinate adjective is to be substituted for that rejected. The most obvious illustration is in predications of place and time: thus we may say 'Mr Smith is not now in this room,' and, knowing that Mr Smith is alive, we know that in the necessities of nature he must be in some other determinate place. Thus we may in a rapid survey discover the *absence* of any object within a given place, independently of any knowledge—by observation or otherwise—of its presence in some other place; and this is sufficient to dispose of the contention that there must be positive ground for a negative judgment. In fact the strictly negative form of judgment is relevant for purposes of further development of thought, whether we are able to assert an opposed positive, or know only that *some* opposed positive could be affirmed if our knowledge were further extended. What is obviously true of time or place predications is also, though not always so obviously, true of qualitative predicates such as colour or tone: for instance, we may deny that a certain sound is that of a piano, because of our familiarity with that instrument, without being able to define the kind of musical instrument from which the sound proceeds, owing perhaps to our unfamiliarity with other instruments; although we may know, first, that it is a musical sound, and secondly on quite general grounds that it must come—not from any instrument whatever—but from some determinate kind of instrument.

§ 2. Having distinguished some of the different ways in which the phrase pure negation may be understood, we will briefly examine the dictum that pure negation has no significance. It may perhaps be at once said that this dictum is itself purely negative, and that therefore

anyone who maintains its significance has committed himself to a contradiction. A more serious treatment of the contention shows that for the word 'significance' we should substitute 'having value' or 'importance' or 'relevance to a specific purpose.' The purpose, for instance, of the above negatively expressed dictum is to oppose some other philosophers who have attributed a false value or importance to the negative judgment. It will be seen that the whole question hinges on the meaning to be attached to the word 'significance.' A form of words may be said to be *absolutely* non-significant when they fail to convey any precise content for thought-construction. This failure of a phrase to convey meaning may be due either to the substantial components themselves or to their mode of combination; thus it is a merely *verbal expression* that may be said to have or not have significance for thought in this absolute sense. But in attributing non-significance to a judgment apart from its verbal expression, the most probable meaning intended is that it does not represent any actual process in thought. But any of the examples taken above go to show that the purely negative judgment cannot be universally charged with non-significance in this sense.

§ 3. We have considered in turn, first the proposition as a whole unanalysed; secondly, the predication of an adjective of a given subject-term; and we now turn to the subject-term itself, apart from the adjective predicated, and raise the question whether any proposition can have significance in case there is no real thing corresponding to the subject-term, although there may be a word or phrase used professedly to denote such thing. Now I have regarded the substantive, which is ultimately

the subject in all propositions, as a determinandum—that is, as something given to be determined in thought; if then there is *nothing* given to be so determined corresponding to the word or phrase by which we intend a certain substantive, then what becomes of the proposition? Consider for example the propositions: 'An integer between 3 and 4 is prime,' and again 'An integer between 3 and 4 is composite.' It must be said that neither of these propositions is true. Now since every integer is either prime or composite, it can be at once seen that any proposition predicating an adjective of the subject 'an integer between 3 and 4' must be false, even though the adjective is appropriate to integers as such. This statement needs only the qualification that we may correctly predicate of an integer between 3 and 4 that it is greater than 3 and less than 4; this, however, is not a genuine proposition but one that is implied in the meaning of the subject-term, and is thus merely verbal. We conclude then that of such a subject-term as 'an integer between 3 and 4' no adjective can be truly predicated in a real or genuine proposition.

We may therefore contrast two cases of a subject-term *S*: (1) where *S* is such that *some* adjective can be truly predicated of it in a genuine proposition, and (2) where *S* is such that *no* adjective can be truly predicated of it in a genuine proposition. These two cases may be briefly expressed—'*S* is' and '*S* is not.' The significance of these two propositions is brought out in considering the process technically known as obversion. The fundamental problem of obversion I will symbolise as the problem of passing from '*S* is-not *P*' to '*S* is non-*P*.' Here, when we hyphen the negative with the

*Obversion*

copula, I understand it to mean that the proposition ' $S$  is  $P$ ' as a unit, is asserted to be false. But when we hyphen the negative with the predicate, we are affirming of the subject  $S$  the kind of predicate called negative; in other words ' $S$  is non- $P$ ' is an affirmative proposition containing a negative predicate, while ' $S$  is not  $P$ ' is a negative proposition in the sense that the attitude of negation applies to the proposition as a whole. Now this transformation from the negative proposition to the positive assertion of a negative predicate, has been assumed as almost trifling, and as only too obvious; but I would wish at once to raise the question as to the condition necessary for the validity of this process, called obversion, in its fundamental form.

As we have already stated, the incomplete proposition ' $S$  is' really means, ' $S$  denotes something of which some adjective may be predicated truly in a proposition not merely verbal.' Thus the scheme by which I express the condition under which obversion is valid, is to add to the explicit negative premiss ' $S$  is-not  $P$ ,' the additional premiss ' $S$  is,' from which we may validly infer the affirmative conclusion ' $S$  is non- $P$ .' The incomplete form of proposition ' $S$  is' means that  $S$  has some character which may be predicated of it, without defining what character can be positively asserted. The conclusion ' $S$  is non- $P$ ' means that we predicate of  $S$  a character, determined so far as that it is an opponent of the proposed character  $P$ , but otherwise indeterminate. An illustration from history will show how this process may be applied. Thus the name *William Tell* is the name of a historical character about whose existence there appears to be doubt. In denying

any proposition which predicates an adjective such as 'submissive' of the subject William Tell, we could not validly predicate of him the contrary adjective 'defiant,' unless we were able first to assert that Tell *is*, in the sense we have explained.

The problem of the *obversion* of a singular proposition is the same as that of formulating accurately the *contradictory* of a singular proposition. Thus, in showing that, in order to pass from the denial (or contradictory) of ' $S$  is  $P$ ' to the affirmation ' $S$  is non- $P$ ' we require the additional datum ' $S$  is,' we have indicated that neither of the propositions ' $S$  is  $P$ ' and ' $S$  is non- $P$ ' would be true, in the case that ' $S$  is' were not true. In other words, the two propositions ' $S$  is  $P$ ' and ' $S$  is non- $P$ ' are not properly contradictories. The contradictory of ' $S$  is  $P$ ' should be formulated in the alternative proposition 'Either  $S$  is-not or  $S$  is non- $P$ '; as also the contradictory of ' $S$  is non- $P$ ' in the alternative proposition 'Either  $S$  is-not or  $S$  is  $P$ .' Thus, in our historical illustration, *neither* of the two propositions 'A certain man named William Tell *submitted to* the Austrians' and 'A certain man named William Tell *defied* the Austrians' would be true, if it were the case that there was no such person as William Tell; and hence the proper contradictories of the two propositions must be respectively expressed in the alternative forms: 'Either there was no such person as Tell or he (Tell) *defied* the Austrians,' and 'Either there was no such person as Tell or he (Tell) *submitted to* the Austrians.'

§ 4. To illustrate the significance of this view we must consider the different types of cases in which a proposition of the form—' $S$  is'—can be truly asserted.

In every case, the term *S* must have sufficiently determinate meaning, to give rise to the alternative propositions '*S* is' or '*S* is not'; the question could not arise if *S* were treated as a mere symbol without significance. When this is agreed, it will be found that any apparent variations in the meaning of the word 'is,' will in reality be variations in the kinds of substantive category to which the name *S* is understood to apply. For instance, let us take the names of substantives under the category of number. We may say on the positive side that the number 3 *is*. This will mean that some true adjectives can be predicated of the number 3, beyond those which might be held as merely involved in the definition or connotation of the word 3; thus, if we should define 3 as meaning  $2 + 1$ , the statement that the number 3 has the characteristic expressed by  $2 + 1$  would be purely verbal. But the number 3, we say, is such that an indefinite number of other adjectives, not included in its definition, can be truly predicated, as for instance that 3 is prime or that 3 is a factor of 12. Contrast the name 3 with the phrase 'an integer between 4 and 5': in the sense in which we can significantly assert that 3 *is*, we may assert that an integer between 4 and 5 *is not*; in other words, no true character can be assigned to this proposed subject, except what is involved in our understanding of its meaning, namely that it belongs to the general category of integer, and that it is to be greater than 4 and less than 5. Generalising from this example, it will be seen that such a subject-term is defined first by reference to a general category (in the above case that of number) and next, by a proposed means of determining or

selecting out of the members of that category, a particular example related in a defined way to other things.

With reference to the category to which the subject-term *S* by definition belongs, any difference of category is naturally associated with an apparent difference in the meaning of 'is.' In particular there is a range of subjects for which the word 'exists' would be naturally substituted for 'is.' Thus it may be agreed that what is manifested in space and time may be said to exist: hence we raise such questions as whether God exists, or whether the centaur Cheiron existed, or whether William Tell existed. The objects intended to be denoted by these subject-terms may be said to belong to the category of the *existent* whether the propositions asserting their existence are true or false. Thus we must maintain, in accordance with the nature of the definition of God, that 'God is an existent,'—this being a merely verbal or analytic proposition; but the question of the truth of the synthetic or real proposition 'God exists' remains problematic. The same holds of Cheiron and William Tell. On the other hand what is denoted by such a subject-term as 3 or an integer between 4 and 5 would not be called an existent. Thus we maintain that there is no difference in the force of the word 'is' in its isolated usage; but that if any difference appears—as when we substitute 'exists' for 'is'—this is merely due to a difference in the category of the subject-term, which again presupposes a difference in the types of adjectives that are properly predicable of it.

When the proposition '*S* is' is under consideration it must be understood that the term *S* is not an ordinary singular name but one of a peculiar nature that has not,

I think, been recognised by logicians. Such a name will be designated by the prefix 'a certain.' Consider the following propositions: 'A certain man was both a philosopher and a historian,' 'a certain integer between 3 and 11 is prime,' 'a certain novel has no hero,' 'a certain flash of lightning was vivid.' The truth or falsity of these propositions could only be decided by the hearer if for the phrase 'a certain' is substituted 'some or other,' 'one or more,' so that for him the reference is indeterminate. Thus, of Hume and of Xenophon it is true that they were both historians and philosophers; between 3 and 11 there are two numbers—5 and 7—that are prime; and the other examples are equally ambiguous. We must suppose that the speaker has in mind a single determinate philosopher-historian, number, novel or flash, which has been identified by him, and to which therefore he may return in thought. From these examples we see that a term may be properly called uniquely singular for the asserter, although in fact there may be several objects answering to its explicit description. Thus from a proposition with the predesignation 'a certain' may be inferred the corresponding proposition with the predesignation 'some or other,' though of course not conversely. This points to two modes in which what is technically called the particular proposition can be inferred: first, from premisses one of which is itself particular; and secondly, from a specific instance for which the predesignation 'a certain' stands.

Now it is the latter form of proposition which raises the problem of the significance of the proposition 'S is.' For the asserter, the contradictory of the proposition that 'A certain man was both an historian and a philo-

sopher' would be that the person of whom he is thinking was not both an historian and a philosopher: or the contradictory of the proposition that a certain integer between 3 and 11 is prime, would be that that same integer is composite; whereas for the hearer, who can only understand the given propositions as particular, the contradictory in the first case would be: 'No man is both an historian and a philosopher' and in the second, 'No integer between 3 and 11 is prime.' In fact, in denying the proposition that a certain integer between 3 and 11 is prime, we must mentally specify the integer about which we are thinking, and assert that this integer is composite. The form of the statement thus reached is equivalent to that of the conclusion in the process of obversion, but it is not obtained here (as in obversion) by the medium of the purely negative premiss, but directly by mentally specifying the number under consideration.

§ 5. It remains to explain more precisely the nature of the denial of 'S is P' which combined with 'S is' yields the conclusion 'S is non-P.' The proposition which merely denies 'S is P' must be understood to involve a hypothetical element. Consider, for example, the statement 'Anyone who calls this afternoon is not to be admitted'; this proposition does not contain any categorical assumption that somebody will call, and may be otherwise expressed in an explicitly hypothetical form 'If anyone calls he is not to be admitted.' Combining this premiss with the further ascertainable fact that a certain person has called, the obvious conclusion, that this person is not to be admitted, follows. In general the symbols that we have used, namely that 'S is P'



is false and that 'S is,' may be explained by making explicit the descriptive, adjectival, or connotative factor in the term symbolised by *S*, which factor we shall symbolise by *M*. The negative premiss then becomes: 'If anything is *M* it will not be *P*': the categorical premiss becomes: 'A certain thing is *M*.' In this formulation the symbol *S* does not appear. Now *S* stands for a certain thing which has not yet been identified and which is only presented in thought by the general description *M*; or briefly *S* is to mean 'a certain thing which is *M*.' In transforming the proposition 'Anything that is *M* will not be *P*' into the form '*S* will not be *P*' we introduce the factor 'a certain thing.' And in transforming the proposition 'A certain given thing is *M*' into the form '*S* is' we have transferred the whole of the adjectival component in the proposition from the predicate to the subject: or otherwise, the two propositions may be rendered 'Anything that may be given having the character *M* will not be *P*' and 'A certain thing having the character *M* is given.' Thus it is not strictly correct to use the same symbol *S* in our two propositions, since the only differentiating element of meaning in the term *S* in the negative premiss is the adjectival or descriptive component, whereas in the categorical premiss the substantival component enters along with the adjectival. It follows that the analysis given is not restricted to the negative form of our first premiss, since the same kind of syllogism would apply to an affirmative conclusion: the essential characteristic of the first premiss is its hypothetical character, as opposed to the other premiss which is categorical. Thus the affirmative case would

be rendered 'Anything that may be given having the character *M* will be *P*,' 'A certain thing with the character *M* is given' therefore 'This thing having the character *M* will be *P*.'

§ 6. In the course of this final explanation it will be noted that for the formula 'S is' in which no determinate adjective is predicated, we might substitute 'S is given' or 'S is real.' Now the words 'given' and 'real' though of course grammatically adjectival, are not in the logical sense adjectival, for their meaning does not contain any indication of character or relation. It may be remarked in passing that the application of the term 'real' includes but goes beyond that of the word 'given.' The postulate that has to be assumed is that, however indeterminately we may have been able to characterise it, the real must have some determinate character. We thus return to our first exposition of the force of the incomplete predication 'S is': namely that *S*, as being real, must have some determinate character although it may be that this character cannot be completely or exactly known by any finite intelligence.



## CHAPTER VI

## THE PROPER NAME AND THE ARTICLES

§ 1. A *simple* proposition '*S* is *P*' involves as subject a single uniquely determined substantive, and as predicate a single uncompound adjective; a *singular* proposition must satisfy the first of these conditions, but its predicate may be simple or compound. Propositions of this nature give rise to the question how the reference in the subject *can* be uniquely determined. Speaking generally, singular names may be divided into two classes according as they contain or do not contain an explicit adjectival or relational component: to the first class belong such terms as 'the smallest planet,' 'the king of England who signed Magna Charta,' 'the cube root of 8'; to the second class 'Mercury,' 'John,' and '2'; and the former will, for convenience, be referred to as descriptive or significant, the latter as proper or non-significant.

Compare now the proper name 'Poincaré' with the descriptive name 'the President of France.' In order that this latter term may have unique application its component 'France' must have unique application: and hence here, as in almost every case, the uniqueness of a descriptive name is only secured through its reference to a proper name. On the other hand we shall find that many so-called proper names contain a descriptive factor: thus the term 'England' contains the termination 'land' which would be normally understood as bringing the

term 'England' within the general class expressed by the word 'land.' This, however, would not explicitly hold of the name 'France'; but, in point of fact, just as England might be taken to *mean* 'the land of the Angles,' so might France be taken to *mean* 'the land of the Franks.' In the same way the terms 'Mr Gladstone' and 'Lord Beaconsfield' have the partial significance expressed by the prefixes 'Mr' and 'Lord' respectively. As another example, the name 'Mont Blanc,' though etymologically equivalent to 'white mountain' and therefore apparently completely significant, must yet be called a proper name since its application is not to *any* white mountain, but to a specific one. We are here in the reverse position to that reached in discussing the term 'England'; for in 'England' we detected the concealed element of significance indicated by the termination 'land,' while in 'Mont Blanc' we have detected the concealed element of non-significance which prevents us from applying the name to *any* white mountain indiscriminately. Now, attributing to the term 'Mont Blanc' the maximum of significance that it can bear, and agreeing that there would be a species of incorrectness in using the term for any object which had not the characteristics 'white' and 'mountain,' yet this admitted significance is not the sufficient ground for applying the term as it is understood by those who use it with a common agreement as to its unique application. We may therefore say that any name which is commonly called a proper name has, so far as our analysis has proceeded, a residual element of non-significance over and above such significance as is naturally recognised in the verbal structure of the name.

§ 2. But if we allow of any name that it contains an element of non-significance, how is it possible that this name should be understood as applying to the same object when used at different times or by different persons or in different and varying connections? Where the name denotes a substantive, the possibility that it should mean the same substantive when used in different propositions, involves the possibility of substantival identification. A similar process, i.e. adjectival identification, is involved when the same adjective is used in different connections. Whether we ask how the substantive-name 'Snowdon,' for instance, can be understood to stand for a definite substantive, or how the adjective-name 'orange' can be understood to stand for a definite adjective, we are in fact confronted with precisely the same logical problem; and hence, if we regard the name 'Snowdon' as a proper substantive-name, we must regard 'orange' as a proper adjective-name. The analogy may be pressed a little further and applied to complex names; for just as an adjective name is exhibited as significant when it is expressed in the form of adjectives combined in certain relations, so a substantive name is exhibited as significant when it is expressed in the form of substantives combined in certain relations. Note the analogy, for example, between the complex adjective-name 'the colour between red and yellow' and the complex substantive-name 'the highest mountain in Wales.' Here the former involves the proper adjective-names 'red' and 'yellow,' just as the latter involves the proper substantive-name 'Wales.' There is, then, in every explication of significance, a residual element in which we reach either a substantive-

name or an adjective-name which can no longer be defined in this form of analysis. And further, the explication of the significance of this residual proper substantive or proper adjective-name involves the conception of identity: in the one case, substantival identity—which is implied when we understand that the substantive denoted by the word 'Snowdon' in one proposition is identical with that denoted by 'Snowdon' in another proposition; and, in the other case, adjectival identity—which is implied when we understand that the adjective denoted by 'orange' in one proposition is identical with the adjective denoted by 'orange' in another proposition.

Our first approximate account of a proper name is then, that the intended application of the given name is to an object—whether it be substantive or adjective—which is identical with the object to which it may have been previously understood as applying in another proposition. For example, to explain what I mean by 'orange' I could say: 'You understand the word colour: and I shall mean by "orange" the colour which you can discern as characterising the object to which I am pointing. And when you identify the colour of any object with the colour of this, its colour is to be called "orange".' The possibility of such appeal presupposes that colour can be perceptually identified in different objects, apart from any other agreements or differences that the objects may manifest. In the same way the explication of a proper substantive-name requires a similar appeal, which assumes the possibility of identifying a concrete object when it may be presented or thought about in different contexts. Thus, if it was asked whom I meant when I talked of Mr Smith, I

might say: 'I mean the man to whom you were introduced yesterday in my study.' The agreement, therefore, which can be maintained in the application of a proper name amongst those who continue to use it with mutual understanding, is secured by what in a quite general way we may call the method of introduction. An object is introduced, and in the introduction a name is given, and when further reference is intended to the same object, the name is repeated which was given in the act of introduction. In this way the nature of the residual element or undefinable factor in adjectives and relations as well as that in the case of proper names is further explained.

x § 3. It is worth pausing here to point out a confusion frequently made in discussing the nature of the proper name. The confusion is that between the cause which has led people to choose one name rather than another name for a given application, with the reason for applying the name—once chosen—to one object rather than to another object. This confusion again can be paralleled in ordinary adjectival names as well as in substantival names: thus, it is one thing to assign the etymological causes of the use of the name 'indigo' rather than some other name to denote a particular colour, and another thing to assign the reason for applying the name, when it has once come into common usage, to one of the colours rather than to some other: the reason for this latter is that the colour presented in a given instance is identical with that to which the name indigo was originally given. This is exactly parallel to the ground on which we should justify our applying the name 'Roger Tichborne' to the man presented in court:

namely, the presumed identity of the man before us with the man to whom his godparents had given the name. Why they chose that name rather than some other name is a matter for historical enquiry. But, to repeat, the etymological or historical account of a name must not for a moment be confused with its significance, or with what, in the case of proper names (substantival or adjectival), takes the place of significance as the condition of mutual understanding.

§ 4. This discussion is closely bound up with the different ways in which the articles, indefinite and definite, are used. The indefinite article in its most general and completely indeterminate meaning is illustrated by such assertions as 'A man must have been in this room,' 'We need a sweep,' 'You ought to make a move with your bishop.' Now if we compare this use of the article with its meaning when it occurs at the beginning of a narrative as for instance: 'Once upon a time there was a boy who bought a beanstalk,' we note an important difference in its significance. In the first set of examples the full significance of the article is made explicit by substituting 'some or other': e.g. 'Some or other man must have been in this room'; in the case of a narrative, where the article prepares the way for future references to period, person or place, it means—not 'some or other'—but 'a certain.' Indeed our story might more logically have begun 'At a certain time a certain boy bought a beanstalk.' When the indefinite article is used in this way to introduce some period, person or place not otherwise indicated, it will henceforward be called the *Introductory Indefinite*, to distinguish it from the *Alternative Indefinite*.

Now suppose the narrative to continue: 'This boy was very lazy'; the phrase 'this boy' means 'the boy just mentioned,' the same boy as was *introduced* to us by means of the *indefinite* article. Here the article 'this,' or the analogous article 'the,' is used in what may be called its *referential* sense. The linguistic condition necessary to render such reference definite is that only *one* object of the class (whether person, period, or place) should have been immediately before mentioned. Other variations of the Referential Definite are such phrases as 'the former' and 'the latter,' which may be required to secure definite reference. The above analysis brings out the necessarily mutual association of the *introductory* use of the *indefinite* article with the *referential* use of the *definite* article. Again, instead of beginning the second sentence with the phrase 'this boy,' language permits us to use a pronoun: thus the word 'he,' in general, is sufficient to denote a specific individual understood by the verbal context; so that here the pronoun serves precisely the same logical function as the referential definite article 'the' or 'this.' A still more important further development of the referential 'the' comes up for consideration when, instead of depending upon immediacy of context—as in the preceding cases of 'this' and 'he'—we refer to an historical personage who has a wide circle of acquaintance as (e.g.) 'The well-known sceptical philosopher of the eighteenth century.' Here the phrase 'the well-known' functions as a referential definite, though there may have been no immediately previous mention of Hume, it being assumed that a certain philosopher will be unambiguously suggested to readers in general, in spite of the fact that there may have been

more than one person answering to the description 'sceptical philosopher of the eighteenth century.' This extended use of the referential definite is quite interestingly illustrated in Greek, where a proper name is prefixed by the definite article 'ὁ'; a usage which appears very happily to bring out the precise function of the proper name, as referring back to an individual who was originally introduced in history or otherwise under that name. The same holds in English of geographical proper names, e.g. the Thames, the Hellespont, the Alps, the Isle of Wight, etc. Lastly, in a narrative, the juxtaposition of a *proper name* with the introductory indefinite supplies a substitute for the referential definite. Thus our story about the beanstalk which begins with the introductory indefinite 'a boy' may be continued either by using the phrase 'this boy'—involving the referential article—or by the pronoun 'he'; or thirdly by the proper name which prepares the way for repeated reference to the same boy: 'Once upon a time there was a boy named Jack who bought a beanstalk.' It will be noted therefore that the way in which the proper name occurs in a narrative where it secures continuity of reference, illustrates the same principle as its use in ordinary intercourse, where it ensures agreement amongst different persons as to its single definite application: in both cases, the understanding of the application of the name involves reference back to the act of introduction, when the name was originally imposed.

There is an important analogy between the singular descriptive name of the kind illustrated by 'the well-known sceptical philosopher of the eighteenth century' and the proper name, in that frequently it is only within

a narrower or wider range of context that the proper name may be said to have a uniquely determined application. Thus, within a family, the name 'John' may be understood to denote one brother of that name; whereas, in a certain period in English history, it will denote the king who signed Magna Charta. That uniqueness of reference is relative to a particular context is similarly seen in such phrases as 'the table,' 'the garden,' 'the river,' which though applicable to different objects in different contexts are understood within a given circle or in a given situation to have a uniquely determined application. The article 'the' used in such cases may be called Indefinite Definite, to distinguish it from the most definite of all uses of the article, namely where the unique application is understood without any limitation of context—in cases, for example, like 'the sun,' 'the earth.'

\* We have thus divided articles (and what are logically equivalent to articles) into four classes: (1) the Indefinite Indefinite, otherwise the Alternative Indefinite; (2) the Definite Indefinite, otherwise the Instantial Indefinite, best expressed by the phrase 'a certain,' which includes the Introductory Indefinite; (3) the Indefinite Definite, otherwise the Contextual Definite, which includes the Referential Definite; and (4) the Definite Definite, for which the understood reference is independent of context.

§ 5. A special form of the contextual definite which is to be distinguished from the referential, is expressed by the terms 'this' and 'that' when used as demonstratives. Literally, the demonstrative method is limited to the act of introducing an object within the scope of

perception. But, when we point with the finger, for instance, to a particular person or mountain or star, our attempt to direct the attention of the hearer to the object intended may or may not succeed: if successful, it will be because there is no other conspicuous object belonging to the class indicated by the use of the general significant name (person, mountain, star, as the case may be) within the range of space to which we have directed attention. The condition for securing unambiguity is not that there should be only one object of the specified class within the range indicated, but that there should be only one such *visible* object; and here observe a parallel between the demonstrative definite, and the case illustrated by the example 'the well-known sceptical philosopher of the eighteenth century.'

§ 6. At this point in our discussion let us consider the special difficulty which attaches to the notion of a proper name. This problem presents a *dilemma*. If we maintain that the proper name is non-significant in some sense, then it would follow that any propositional phrase that might contain the proper name would be non-significant in the same sense. If, on the other hand, we attempt to assign some definite significance to the proper name, this will entail our substituting a uniquely descriptive name as equivalent in meaning to the proper name, in which case the distinction between the descriptive name and the proper name would vanish.

This problem raises a question relating to the wider problem of the definition of words or phrases. Taking the two words 'valour' and 'courage,' the brief formula 'valour means courage' is seen on reflection to be imperfectly expressed. Everybody would agree that what



is intended here is that the two terms *valour* and *courage* have the same meaning; i.e. that the quality meant by the one term is the same as the quality meant by the other. Hence a more correct expression than 'valour means courage' would be 'the word valour means what is meant by the word courage.' Where a phrase instead of a single word is under consideration the same principle is involved. For example, '*p* is a factor of *q*' means-what-is-meant-by '*q* is divisible by *p*'; or again 'some benefactor of *A*' means-what-is-meant-by 'one or other person who has benefited *A*.' These illustrations bring out the distinction between (a) the relation which one word or phrase may bear to another word or phrase, and (b) the relation which a word or phrase may bear to what is called its 'meaning.'

Now the propositions which allow us to substitute one phrase for another may be called bi-verbal definitions<sup>1</sup>; and the relation that is to be affirmed as holding between two such phrases must be expressed in the complex form 'means what is meant by,' or even—when we distinguish between the phrase which has not been understood and that which has been understood—in the still more complicated form 'is to be understood to mean what has been understood to be meant by.' This last complication brings out the purpose that a definition has always to serve; namely the elucidation of a phrase assumed to require explanation in terms of a phrase presumed to be understood.

<sup>1</sup> It has been suggested that a more correct substitute for 'bi-verbal definition' would be 'translation.' But whichever terminology is employed, the distinction between the kind of definition called *translation* and some more ultimate definition remains.

This formulation of the bi-verbal definition leads us to consider what, in contrast, we shall call the uni-verbal definition. When we speak of a phrase as being 'already understood,' it is equivalent to saying that the *meaning* of the phrase is known. The formula that 'phrase *p* means what is meant by phrase *q*,' in short, raises the question, What is it that phrase *q* means? Let us first consider the *kind* of entity that a phrase could mean. Phrases which would be grammatically or logically distinguished in regard to type or category would *mean* entities belonging to correspondingly different types or categories: thus one phrase would mean a certain proposition; another would mean a certain adjective, another a certain substantive, and so on. Thus 'courage' means a certain adjective or quality-of-conduct, 'horse' means a certain substantive or kind-of-animal. A phrase prefixed by an article such as a, the, some, every, any, requires special consideration. Thus, if we were to substitute for such phrase any phrase that means what is meant by the given phrase, the article or some equivalent would still remain. Thus 'the first novel of which Scott was the author' means what is meant by 'the romance that was written by Scott before any other of his romances.' In this bi-verbal substitution the word 'the' is retained. Now consider, in contrast to the proposition stating the equivalence in meaning of the above phrases, the proposition 'The first novel written by Scott was called *Waverley*'; or, inasmuch as there is only one novel that is known bearing this name, we may put the statement in the form: 'The first novel written by Scott was the novel called *Waverley*.' Such a proposition is of nearly the same



type as 'The author of *Waverley* was the author of *Marmion*.' In both of these propositions the relation of identity is asserted in regard to two uniquely descriptive terms. But neither of these propositions is verbal; in neither case could we substitute for the relation of identity the expression 'means what is meant by.' Hence we are not identifying the *meaning* of the two phrases: i.e. we are not identifying what is meant by one phrase with what is meant by the other. What then is it that we are identifying? In the language of Mill we should say we are identifying what is *denoted*—and in the language of Frege what is indicated or (as we prefer to say) factually indicated—by the one phrase with what is denoted or factually indicated by the other. Now, as our term suggests, an appeal to *fact* is required in order to understand what it is that is *factually indicated* in distinction from what is *meant* by a certain phrase. Hence, though a knowledge of the usage of language alone is sufficient to know what a phrase means, a knowledge of something more than mere linguistic usage is required to know what a phrase denotes or factually indicates, whenever we are dealing with a phrase that indicates something different from what it means. The word 'courage' or the phrase 'not flinching from danger' is of such a nature that there is no distinction between what it means and what it indicates or denotes; it is only phrases prefixed by an article or similar term for which the distinction between meaning and indication arises. Turn now to the peculiarities of the illustration given above: 'The first novel written by Scott was the novel called *Waverley*.' The interpretation of this statement is that the object indicated

by the phrase that stands first is the same as that indicated by the phrase 'the novel called *Waverley*' although the meanings of the two phrases differ. Take a parallel case: 'The colour of the object at which I am pointing is identical with the colour that is called red'; here again the identity of what is *indicated* by the two phrases does not carry with it identity in what is *meant* by the two phrases. In short, where we have an identification of what is indicated in spite of non-identity in what is meant, we recognise that the statement of identity is not merely verbal but factual.

In the above illustrations we have taken such names as Scott and *Waverley* to exemplify names universally recognised as proper; while the phrase 'the first novel written by Scott'—or any phrase having the same meaning—would be called descriptive in a sense primarily intended as antithetical to proper. Now one step was taken to bridge this antithesis when we used the proper name in the extended phrase 'the novel called *Waverley*': i.e. the single name *Waverley* is a proper name and the compound phrase 'the novel called *Waverley*' is constructed in the form of a descriptive name. We are thus leading up to the view that what is *indicated* by the descriptive phrase—'the novel called *Waverley*'—is identical with what is *meant* by the proper name 'Waverley.' Thus, in interpreting the simple proposition '*Waverley* was the first novel written by Scott,' which is recognised at once to be factual not verbal, we are identifying what is factually indicated by the subject and predicate terms respectively; and in the case of the proper name 'Waverley,' what it factually indicates is indistinguishable from what it means. Hence it seems

Proper Name  
Defined:

legitimate or possible to define a proper name as a name which *means* the same as what it *factually indicates*.

§7. We may now introduce the technical term 'ostensive' which will suggest as its opposite the familiar term 'intensive.' A proper name may be said to be ostensively definable in contrast to those more ordinary terms which are said to be intensively definable. This ostensive definition will be only a special instance of a form of definition involving the complex relation 'means what is indicated by'—a relation which is involved in any attempt to define a proper name by means of a descriptive name. The particular force of the notion of ostensive definition will now be explained, and it will be found to apply both to an adjectival and to a substantival name. The ordinary proper name applies to an object whose existence extends over some period of time and generally throughout some region of space. The appearance of such an object in perception (or rather of some spatially or temporally limited part of that object) provides the necessary condition for imposing a name in the act of indicating, presenting or introducing the object to which the name is to apply, and this it is that constitutes ostensive definition. In extending the notion of a proper name to certain adjectives our justification is that ultimately a simple adjective-name—such as red—cannot be defined analytically but only ostensively. Theoretically, we must suppose that any name, singular or general, proper or descriptive, substantival or adjectival, has originally been imposed on a particular occasion by a particular person or group of persons. In the case of ostensively defined names, the occasion on which definition is pos-

sible must be one on which the object is actually presented. When, however, the meaning or application of such a name has afterwards to be explained, or so-to-speak redefined, the only direct method is to secure for the enquirer another presentation of the object in question. Thus John Smith, having been presented to his family at birth—which we may take to be the occasion on which the name was imposed—must be presented again to the person ignorant of its application. Hence, in introducing a man under the name John Smith, we are using the same ostensive method as was required in the original definition, but such mention of the name does not, properly speaking, constitute definition. We are stating, in effect, the proposition—which is not merely verbal—that 'the person introduced is identical with the person upon whom the name was originally imposed.' This case of an ordinary substantival proper name is analogous to that of an adjectival name—say cochineal—which originally could only have been ostensively defined, and which must therefore be ostensively redefined for the person ignorant of its application, in the form 'the colour of this presented object is identical with that upon which the name cochineal was originally imposed'—a statement which again is not merely verbal. When a proper name is called arbitrary, this arbitrariness attaches only to the original act of imposition; but, when the application of the name is afterwards explained, such explanation is no longer arbitrary, since to be correct the *real* proposition that the substantive or adjective presented is identical with that upon which the name was originally imposed, must hold good, and this statement may be either true or

false, apart from linguistic convention. Furthermore, when ostensive definition is employed, it must be observed that we do not say that the proper name means what is *meant* by such a phrase as 'the object to which I am pointing' (which after all is only an instance of a descriptive phrase), but we say that the proper name means what is *indicated* by the descriptive phrase 'the object to which I am pointing.' For it is obvious in this case, as in the more general account of a descriptive phrase, that however we may further explicate the *meaning* of the phrase 'the object to which I am pointing,' the substituted phrase would not have the nature of a proper name but necessarily of a descriptive name.

When, then, finally we agree with the general position of the best logicians that the proper name (as Mill says) is non-connotative, this does not amount to saying that the proper name is non-significant or has no meaning; rather we find, negatively, that the proper name does not mean the same as anything that could be *meant* by a descriptive or connotative phrase; and positively, that it does precisely *mean* what could be *indicated* by some appropriate descriptive phrase. This exposition holds both for the names of objects which can be presented and thus ostensively defined; and also for the names of objects removed in time or place, for the definition of which a descriptive phrase (which is other than ostensive) must be employed.

ut in p. 80, h  
says - 'proper  
non-significant'  
1 of p. 99.

## CHAPTER VII

## GENERAL NAMES; DEFINITION AND ANALYSIS

§ 1. HAVING, in the preceding chapter, distinguished the different kinds of articles, we now turn to a common characteristic in the use of an article, namely, its attachment to a *general* name. The general name has usually been differentiated by reference to number, and roughly defined as a name predicable of more than one object. In fact, however, there are general names such as 'integer between 3 and 4' or 'snake in Ireland' that are predicable of *no* object, while 'integer between 3 and 5' and 'pole-star' are general names predicable of only *one* object. There is therefore nothing in the *meaning* of a general name which could determine the number of objects to which it is applicable. Rejecting this reference to number, we may point out that a universal characteristic of the general name is its connection with the article—the use of the grammatical term 'article' being extended to include this, that, some, every, any, etc. All terms of this kind serve to determine the intended application of reference in a proposition, and hence might more properly be called *applicatives* or *selectives*. Now a general name is distinguished as that to which any applicative can be significantly prefixed: thus the applicative, on the one hand, requires a general name, while, on the other hand, it follows from the essence of the general name that to it any applicative is

significantly attachable. And a further and connected characteristic of the general name is that it can always be used in the plural, or, in fact, with any numerical prefix.

If now we further consider the typical applicatives 'every' and 'some,' we find that in the last analysis they entail the conception of unlimited generality, and language reflects this unlimited reference by uniting these applicatives with the substantival term 'thing' to form the single words 'everything' and 'something,' where the word 'thing' stands only for abstract generality without any such limitation as could possibly be defined by adjectival characterisation. If any account of the name 'thing' here could be given in philosophical terminology, we should have to say that 'thing' stands for the category of all categories; or more precisely, that the generality implied in this use of the word 'thing' is so absolute that it must include all logical categories, such as proposition, adjective, substantive, etc. On the other hand, the general term with which ordinary logic is almost exclusively concerned, is removed by two steps from this absolute generality: for example, the general term 'man' comes under the logical category 'substantive,' which itself is a limitation of the absolute generality peculiar to the word 'thing.' Thus in philosophical logic we ought to be prepared to define the category 'substantive' by some limiting characterisation within the absolute general 'thing'; and similarly of the category 'adjective' or 'proposition,' etc. And again, in ordinary logic, any general name is understood to be defined first, as coming under an understood category—in particular the category sub-

stantive—and secondly as delimited by a certain adjectival characterisation. As regards any specified general substantive-name, its nature as general is brought out by showing its meaning to be resolvable into some specific adjective or conjunction of adjectives, juxtaposed to the category 'substantive' itself.

The consideration that to the general name any applicative can be prefixed, distinguishes it from the singular name, whether descriptive or proper. We have now to bring out a characteristic shared in common by the general name and the singular descriptive name, which distinguishes them both from the proper name: namely that the two former always (except in the case of the absolute general name 'thing') contain in their meaning an adjectival factor, whereas the strictly proper name contains in its meaning no adjectival factor. This account may seem a somewhat arbitrary way of settling the prolonged controversy initiated by Mill as to whether proper names are non-connotative. In default of a definition of a proper name, however, it is impossible to decide whether any given name, such as London, is to be called proper. I propose, therefore, to define the word proper as equivalent to non-connotative, non-descriptive or non-significant (since these three terms are themselves synonymous), and the only debatable point which remains is as to whether any names can properly be called 'proper'. A similar remark applies to the question whether all general names are connotative, since it would seem necessary to define a general name

<sup>1</sup> This brief account of the Proper Name has been discussed and qualified in the preceding chapter; but for the present purpose it is adequate.

(except the absolutely general name 'thing') as equivalent to one which is connotative.

§ 2. At this point we must enquire into the precise meaning of the word *connotation*, and this enquiry necessitates the introduction of the more general notions of intension and extension. While extension stands for a set of substantives, intension stands for a set of adjectives; and moreover the two terms are used in correlation with one another, the substantives comprised in the extension being characterised by the adjectives comprised in the intension. Now of all the adjectives that may characterise each of a set of substantives, a certain sub-set will have been used for determining the application of the general name, and the range of extension thereby determined constitutes the denotation of that name. Thus the specific function of connotation is that it is used to determine denotation; and hence any other adjectives that may characterise all the substantives comprised in the denotation, do not determine the denotation, but rather are determined by it. The entire and often innumerable conjunction of adjectives determined by the denotation has been called by Dr Keynes, the *Comprehension*. Thus Dr Keynes's exposition may be summed up in the statement that connotation in the first instance determines denotation, which in its turn determines comprehension. The controversy on the question of what adjectives should be included in the connotation of the general term, has arisen from the false supposition that the logician starts with a known range of denotation, and that with this datum he has to discover, amongst the known common characters peculiar to the members of the class, those which shall constitute the

connotation. But this is reversing the order of priority, since there is no means of delimiting the range of denotation of a term otherwise than by laying down a finite enumeration of adjectives which, taken together, constitute the test for applying the name. This test, which is provided by the connotation, may have remained unchanged, or have varied in the course of time, or again have been technically determined by science in such a way as to conflict more or less with common usage; but in every case what constitutes the connotation is invariably a conjunction of adjectives which, within these limits of time or context, is *used* to determine the application of the name.

We are indebted to Mill for an elaborate and complete treatment of connotative names, including the connection between connotation and definition; but the formula that the definition of a general name is the unfolding of its connotation, must be corrected by the added reference to the substantival element, since connotation by itself is purely adjectival. It is obviously impermissible, for instance, to substitute for the substantival name 'a man' a mere adjective 'human' or the abstract term 'humanity'; rather 'a man' should be defined as 'a human being' or 'a being characterised by the attribute humanity.' In other words, a substantive-name must be so defined as to show that it is substantival. The word 'being' here has the force of the word 'substantive,' so that pushed to its logical conclusion, 'a man' should be defined as 'a human substantive'; and it is the *adjective* 'human' which alone may require further analysis in our definition, not the word 'being' or 'substantive.'



The same principle will apply to the definition of a general adjectival name, if it is admitted that certain adjectives and relations (which may be called secondary) can be properly predicated of other adjectives (which in this connection might be called primary). For example, in the proposition: 'the unpunctuality of his arrival was annoying,' we appear to be predicating of the primary quality or adjective represented by the abstract name 'unpunctuality' the further or secondary adjective 'annoying.' By a secondary adjective we mean an adjective of an adjective, and if the primary adjective is expressed grammatically as a substantive, the secondary adjective would be expressed grammatically as an adjective, but if the primary adjective retains its grammatical form as an adjective, then its secondary adjective is expressed by an adverb, which is logically equivalent to an adjective of an adjective. Thus the following series of propositions: '*A* is moving,' 'the movement of *A* is rapid,' 'the rapidity of the movement of *A* is surprising'—involve the primary adjective 'moving,' the secondary adjective 'rapid,' and what we must here call the tertiary adjective 'surprising.' When the primary adjective 'moving' retains its adjectival form, the secondary adjective predicated of it would be expressed as an adverb; thus '*A* is moving rapidly'; or when the secondary adjective 'rapid' retains its adjectival form, the tertiary adjective predicated of it is again expressed as an adverb, as in '*A*'s movement was surprisingly rapid.' These examples seem to confirm the view that adjectives can properly be predicated of adjectives as such. If this analysis is correct, we should expect to find certain general adjectival names (ana-

logous to general substantival names) whose meaning could be elicited in terms of the secondary adjectives implied in their use. Thus to take Mill's example, the name 'fault' which is predicable of such primary qualities or adjectives as 'laziness,' 'unpunctuality,' 'untidiness,' is predicable of such qualities of conduct on the ground that these are characterised by the further or secondary quality 'faultiness.' In this way the definition of a *general* adjectival name would be formulated in terms of the secondary adjective or conjunction of adjectives, constituting its connotation, juxtaposed to the category adjective itself.

§ 3. Turning now to another aspect of our topic, we shall consider the nature of definition only in the form of bi-verbal substitution, where the reference is restricted to words or phrases; in contradistinction to ideas or things, which some philosophers have undertaken to define. The question of the definition of words requires a wider treatment than that generally accorded to it in logic; for under the influence, I presume, of scholastic doctrine, definition has been tacitly restricted to the case of substantive terms, to which the traditional formula 'per differentiam et genus' is alone applicable. It seems, indeed, as if logicians had shrunk in terror from the task of defining other than substantive-terms, setting aside preposition words, conjunction words, pronominal words and even adjectives, as if these, as such, were outside the scope of logical definition. The problem of definition, it is clear, must extend to *any* word, however it may be classified by grammar. We must certainly come to some mutual understanding of the meanings of such words as 'and,' 'or,' 'if,' inasmuch as

Definition:

x



these and many other such words have a meaning, the understanding of which is essential in the pursuit of logic. It is frequently impossible, however, to define a word taken in isolation, and in such cases, we must construct, as that which is to be directly defined, a verbal phrase containing the word. We should not perhaps go far wrong if in every such case we took the completed *propositional* phrase as that which is to be defined, although it is often possible of course to take only some part of a proposition and succeed in meeting the requirements. We repeat then that our problem is how to define a given verbal phrase; and the answer is to substitute for it another verbal phrase. This is the complete and quite universal account of the *procedure* of definition, which justifies our restriction of the topic to bi-verbal definition; its obvious *purpose* is fulfilled if the substituted phrase is understood. (Cp. preceding chapter.)

In this connection it is worth noting that, when what has to be defined is a verbal phrase rather than a single word, we may italicise—so to speak—that part of the phrase for which an explanation is asked. In such cases the remaining components of the phrase may be, and generally ought to be, repeated in the phrase constituting the definition. In symbols, let us say that the phrase '*abcd*' requires definition, where the components '*bc*' are combined in the whole '*abcd*' in such way that the combination '*abcd*' is not understood. Suppose the symbol '*apqd*' to represent our definition; then we shall have defined '*abcd*' (where the component '*bc*' required explanation), by the phrase '*apqd*' in which '*pq*' explicitly replaces '*bc*,' and is offered in expla-

nation. A definition such as that symbolised above is rejected in current logical text-books on the ground that it commits the sin of tautology; for it repeats verbatim the symbol '*ad*' in the definition given as explanation of '*abcd*.' But this mode of definition, so far from being a ground of condemnation, exactly answers in the most adequate sense the requirements. The more exactly we repeat in our definition the actual words and their form of combination, used in the phrase to be explained, the more precisely do we meet the demands for an explanation. Typical instances of this principle will occur to anyone who reflects on the subject. It follows that no general or merely formal criticism of a definition can be made by any logical rule; the question whether any proposed definition is good or not being entirely relative to the enquirer's knowledge and ignorance of meanings. This by no means precludes the possibility of definitions which would be generally useful, because any obscurity or ambiguity which one person might feel is likely to be felt by others; a little common sense is in general all that is necessary.

This account leads at once to one conclusion, which is perhaps tacitly understood by all logicians and philosophers; i.e., that, inasmuch as the only way to explain one verbal phrase is to substitute another, therefore no successive chain of explanatory phrases can serve the purpose of ultimate explanation, if that chain is endless. Hence perhaps the important point in the theory of explanatory definition is that it must stop. In other words, by a shorter or longer process, every definition must end with the indefinable. A certain misunderstanding as to what in logic is meant by the indefinable

must here be removed; for it has been frequently supposed that the indefinable means that which is admittedly not understood. But so far from meaning the 'not-understood,' the indefinable means that which *is* understood; and philosophy or logic may ultimately adopt a term as indefinable only where, because it is understood, it does not require a further process of definition. Philosophy must never stop with the indefinable, in the sense of reaching a component of thought expressed obscurely or with an admitted margin of doubt as to meaning. The indefinable does not therefore mean that which is presented as having no understood meaning, but that whose meaning is so directly and universally understood, that it would be mere intellectual dishonesty to ask for further definition.

§ 4. There has been practical unanimity in regarding definition as a process which essentially involves analysis. We have above reached the idea of an indefinable; and it has been almost universal, I believe, to regard the indefinable as equivalent to what is incapable of analysis—a view which is obvious if definition and analysis coincide. But, if definition is not to be merely equated to analysis, then we may pause before we regard anything as being indefinable on the ground of its being, in some proper sense, unanalysable. In my own view, definition assumes so many varied forms that its equivalence to analysis seems to be highly dubious.

Before entering into further detail on this point it will be well to consider what is meant by the word *analysis* in philosophy and logic. Associated with it we often find a reference to parts and wholes: thus, analysis is

often said to mean the separation of a whole into its parts. But 'separation' does not adequately represent the process. For instance, grammatical analysis does not mean taking the several words of a sentence and determining the part of speech and inflection, etc., of each word, but its object is to show how the significance of the sentence is determined by the mode in which the several words are combined. Similarly the analysis of a psychosis is not merely the cataloguing of such elements as knowing, feeling, acting, but rather the representation of the essential nature of a given psychosis as determined by the mode in which these factors combine. What holds of grammatical and psychological analysis holds of every kind of analysis; and, since the important process is—not the mere revelation of the parts contained—but rather the indication of their mode of combination within the whole, analysis is better defined as the exhibition of a given object in the form of a synthesis of parts into a whole. In this way we can say that any process of analysis can also be described as a process of synthesis; but this does not amount to saying that analysis means the same as synthesis, any more than that the relation 'grandfather' is the same as the relation 'grandson,' although the fact that *A* is the grandfather of *B* is the same fact as that *B* is the grandson of *A*. In short, analysis is the inverse of synthesis; i.e. when the whole *X* is analysed into its several components *a, b, c, d*; then *a, b, c, d* have to *X* the inverse relation which *X* has to *a, b, c* and *d*. In this way it is clear that to analyse *X* simply means the same as to exhibit *X* as a synthesis.

Now instead of taking *X* as a term to be defined,

and exhibiting it as a synthesis of  $a$ ,  $b$ ,  $c$  and  $d$ , let us take the term  $a$  and define it by showing how it functions in a whole  $X$  where it is combined with  $b$ ,  $c$  and  $d$ . This points to two modes of definition, viz. analytic and synthetic. In analytic definition we pass from an unanalysed—i.e. an apparently simple—to an analysed equivalent; in synthetic definition we exhibit the nature of what is simple, not by representing it as a complex, but by bringing it into synthetic connection in a complex of which it is a component. Or more shortly: analytic definition is explaining a complex in terms of its components, and synthetic definition is explaining components in terms of a complex. A few illustrations will help to make clear exactly what we mean by synthetic definition. Take, in arithmetic, the definition of 'factor' or 'multiple.' We first construct a certain complex, involving integers illustratively symbolised by  $a$ ,  $b$ ,  $c$ ;—namely, the proposition  $a \times b = c$ ; this complex, *being understood*, is used to define the terms that require definition, and the definition assumes the following form:  $a$  is said to be a factor of  $c$ , or  $c$  is said to be a multiple of  $a$ , when the relation expressed in the proposition  $a \times b = c$  holds. Here we do not resolve the meanings of the terms 'factor' or 'multiple' into their simple components of meaning, but define them by showing in what way they enter as components into the understood construct—viz. the particular equational proposition. Again taking the algebraic definition of 'logarithm': we begin by constructing a certain whole—expressed for convenience in terms of general illustrative symbols  $b$ ,  $l$ ,  $p$ ,—viz. the proposition that ' $b$  to the power  $l$  equals  $p$ .' This constructed complex—

the nature of which is presumed to be understood—is the medium in terms of which the notion of a logarithm will be defined as follows: ' $l$  is the logarithm of  $p$  to the base  $b$ ' means what is meant by the proposition: ' $b$  to the power  $l$  equals  $p$ .' Again take the definition of 'sine': here, starting with an angle  $A$  bounded at the point  $O$  by the lines  $OX$ ,  $OR$ , we make the following construction; taking some point  $P$  in  $OR$ , and dropping the perpendicular  $PM$  upon  $OX$ , then the sine of  $A$  means the ratio of  $MP$  to  $OP$ . The constructed complex here, namely of a certain right-angled triangle, has first to be indicated and understood, and by its means we are enabled to define what may be called a component of this complex. Another example of this form of definition is afforded in any attempt to define the words 'substantive' and 'adjective.' Here we may presuppose that the notion of 'proposition' is understood—e.g. as that of which 'true' or 'false' may be significantly predicated—and further presupposing that the notion of characterisation is understood, the first account of substantive and adjective will be that they are combined in a proposition in a mode expressible either in the form that 'a certain substantive is characterised by a certain adjective,' or that 'a certain adjective characterises a certain substantive.' Here we give at the same time the definitions of substantive and of adjective by showing how, as components in the whole, i.e. the proposition, they have to be combined<sup>1</sup>.

#### § 5. The general notions of analysis and synthesis

<sup>1</sup> All the definitions occurring in a symbolic system, whether Logical or Mathematical, should in my view be synthetic (in the above sense) and never analytic.

are often explained in terms of parts and whole; but these latter terms should be used for a process essentially different from analysis. At least three processes which are commonly confused are here to be carefully distinguished, viz., partition, resolution, and analysis proper; probably other more subtle variations must finally be recognised.

- (i) By *partition* is meant transforming what is first presented as a mere unit by exhibiting it in the form of a whole consisting of parts; it is perhaps more generally defined as the process of dividing a whole into its part; but it is of the first importance to point out that until a thing is presented as having parts, it cannot be said to be a whole. This conception of part and whole should be strictly limited to three types of cases: (1) to an aggregate, and to a number as the adjective of an aggregate; (2) to what occupies space and to the space which it occupies; (3) to what fills time and to the time filled. These three cases bring out the essential nature of the conception, viz., that the parts must always be conceived as homogeneous with one another and with the whole which they constitute; and further that a certain character called magnitude is predicable of any whole, the measure of which is equal to the sum of the magnitudes predicable of the parts<sup>1</sup>.

- (ii) Next, consider the term *resolution*. This is very shortly explained as the process of exhibiting a composite in terms of its components; but such a definition is open to the same kind of criticism as we have levelled

<sup>1</sup> No intensive or qualitative characteristic of an object can be regarded as a 'whole' of which a magnitude can be predicated by addition of the magnitudes of its 'parts.'

against the popular definition of partition, and by a parallel emendation we shall say that resolution means the exhibition of what is presented as simple in the form of a composite of which the components are assigned. An example of psychological interest is the resolution of a chord heard into its component notes, or again of a note into its component tones, where in either case the combination describes or accounts for the sound as heard. While, on Helmholtz's theory of auditory sensations, the physiological process here involved would be represented as a whole capable of partition, it remains none the less true that psychological apprehension presents the sound as a composite to be resolved.

Then thirdly, I should restrict the word *analysis* to (iii) a process which is distinctively *logical*, and which assumes its simplest form when we combine various adjectives as predicable of one and the same substantive, by means of the mere conjunction 'and.' A simple example will bring out the distinction between resolution and analysis. We have shown what is meant by resolving a note into its component tones. Now the character of the note is described under certain aspects, such as pitch, intensity and timbre, and this constitutes an *analysis* of its character. These three characteristics are predicated of the sound, not in the sense of resolving the sound into various component sounds, but in the sense of characterising the sound itself, whether it be composite or simple. Thus taking timbre for instance as one of the constituent characters, if a note contains three partial tones this would count as 3 in its resolution but 1 only in its analysis; if on the other hand the note were simple—i.e. contained only one

component tone—this would count as 1 in the resolution, while the analysis of this single note yields the same 3 characters (pitch, timbre and intensity) as that of the composite note. The value of this illustration is that it conclusively disposes of the assumption that a plurality of predications characterising an object depends at all upon its partition or resolution; that is, upon regarding the object either as a whole consisting of parts or as a composite resolvable into components.

So far we have considered that form of analysis which exhibits its object as a synthesis of constituents conjoined by the conjunction 'and,' and yielding what will be termed a compound. In contrast to a compound synthesis we must consider also what must be called a complex synthesis—namely one in which the material constituents are heterogeneous, including substantives of different kinds and adjectives of different order—monadic, diadic, triadic<sup>1</sup>—which, *qua* heterogeneous, are united in different modes from that of simple conjunction. Thus the word 'courageous' yields the complex synthesis 'not flinching from danger'; where the material constituents are 'danger' and 'flinching from,' of which the former is expressed substantivally and the latter as a diadic adjective. From this fairly simple example it will be seen that the possible forms of complexity that analysis may yield are inexhaustible. The discussion of this topic will be continued from a somewhat different aspect in a subsequent chapter on relations.

<sup>1</sup> See Chapter IX.

## CHAPTER VIII

### ENUMERATIONS AND CLASSES

§ 1. AN enumeration is an assignment of certain items which may be said to be *comprised* in the enumeration. We attach therefore to an enumeration the conception of unity as applied to the whole along with plurality of the items comprised in this whole. In naming the items to be comprised in an enumeration as *a, b, c, d, e*, for instance, it will generally be implied that we shall not repeat any item previously named; also, that the order of assignment is indifferent. For the purposes of elementary illustration, we shall consider that *all* the items ultimately to be enumerated have a finitely assigned number (say) 12: which may be named respectively *a, b, c, d, e, f, g, h, i, j, k, l, m, n*. Such an enumeration might be called our enumerative *universe*. Thus taking any 'assigned enumeration *included in this universe*, we may speak of the *remainder* to this enumeration—by which will be meant the items comprised in the universe, but *not* comprised in the first assigned enumeration. The notion of *remainder* is therefore associated with the notion of *not*; although the two must be strictly distinguished. The remainder to an assigned enumeration is the simplest function of a single enumeration with which we shall be concerned. We next consider the typical functions of two enumerations—*E, F* say:

*But see p. 118  
for an inverse  
statement*



namely ' $E$  into  $F$ ,' and ' $E$  with  $F$ .' By the former is meant the largest enumeration which is included both in  $E$  and  $F$ ; by the latter, the smallest enumeration that includes both  $E$  and  $F$ . Thus, let  $E$  be  $[a, b, c, d, e, f]$  and let  $F$  be  $[d, e, f, g, h]$ ; then ' $E$  into  $F$ ' will be  $[d, e, f]$ ; and ' $E$  with  $F$ ' will be  $[a, b, c, d, e, f, g, h]$ . Anticipating elementary arithmetical notions, we may at once assert the generalisations: first, that the number for ' $E$  into  $F$ ' cannot be greater than the number for  $E$  or the number for  $F$ ; and secondly, that the number for ' $E$  with  $F$ ' cannot be less than the number for  $E$  or the number for  $F$ . If  $E$  and  $F$  are identical, then in every sense of the word equals,  $E$  into  $F = E$  with  $F = E = F$ . This represents one limiting case. If, on the other hand, some items comprised in  $E$  are comprised in  $F$ , the number for ' $E$  with  $F$ ' is less than the sum of the numbers for  $E$  and  $F$  respectively, and if there are no items comprised both in  $E$  and in  $F$ , then the number for ' $E$  with  $F$ ' will equal the sum of the numbers for  $E$  and  $F$  respectively, while the number for ' $E$  into  $F$ ' is zero. In the former case  $E$  and  $F$  would be said to be *not exclusive*, and, in the latter, *exclusive* of one another. For the general case:

The number for  $(E \text{ into } F) + \text{the number for } (E \text{ with } F)$   
 $= \text{the number for } E + \text{the number for } F.$

For the purpose of further development, we will abbreviate the term remainder into  $R$ . The symbol  $R$ , i.e., *remainder to*, and the prepositions *with*, *into* may be called 'operators,' because each indicates a certain operation to be performed upon one or upon two enumerations by means of which another related single enumeration of the same order is to be constructed.

The relation of the given operation to the enumeration which is to be constructed will be called the relation of *yielding*. Now, noting that  $R'[a, b, c, d, e, f]$  yields  $[g, h, k, l, m, n]$  and that  $R'[d, e, f, g, h]$  yields  $[a, b, c, k, l, m, n]$ , we may illustrate the import of these operators in the eight following formulations where the symbol  $=$  stands for *yields*:

- (1)  $[a, b, c, d, e, f] \text{ into } [d, e, f, g, h] = [d, e, f]$
- (2)  $R'[a, b, c, d, e, f] \text{ into } R'[d, e, f, g, h] = [k, l, m, n]$
- (3)  $R'[a, b, c, d, e, f] \text{ into } [d, e, f, g, h] = [g, h]$
- (4)  $[a, b, c, d, e, f] \text{ into } R'[d, e, f, g, h] = [a, b, c]$

where (1) with (2) with (3) with (4) = the enumerative universe.

- (5)  $R'[a, b, c, d, e, f] \text{ with } R'[d, e, f, g, h] = [a, b, c, h, k, l, m, n]$
- (6)  $[a, b, c, d, e, f] \text{ with } [d, e, f, g, h] = [a, b, c, d, e, f, g, h]$
- (7)  $[a, b, c, d, e, f] \text{ with } R'[d, e, f, g, h] = [a, b, c, d, e, f, k, l, m, n]$
- (8)  $R'[a, b, c, d, e, f] \text{ with } [d, e, f, g, h] = [d, e, f, g, h, k, l, m, n]$

where (5) into (6) into (7) into (8) = the enumerative zero.

As regards these eight formulae we observe that each of the pairs (1) and (5), (2) and (6), (3) and (7), and (4) and (8) give two enumerations related the one to the other as remainder. What holds in one illustration can be formulated in general terms. Let  $E$  and  $F$  be any two enumerations, then: the operation ' $E$  into  $F$ ' yields-the-enumeration-yielded-by the operation  $R$  ( $R'E$  with  $R'F$ ). Since the relation 'remainder to' as also the relation 'yields-what-is-yielded-by' are reversible or symmetrical, this single formula includes all the eight formulae which have been illustrated above. But, we may for the sake of emphasis, express the principle again in eight formulations where  $=$  will now stand for 'yields-what-is-yielded-by.'



- (1)  $E$  into  $F$  = the remainder to  $R^0E$  with  $R^0F$   
 (2)  $R^0E$  into  $R^0F$  = " " "  $E$  with  $F$   
 (3)  $R^0E$  into  $F$  = " " "  $E$  with  $R^0F$   
 (4)  $E$  into  $R^0F$  = " " "  $R^0E$  with  $F$

where (1) with (2) with (3) with (4) = the enumerative universe.

- (5)  $R^0E$  with  $R^0F$  = " " "  $E$  into  $F$   
 (6)  $E$  with  $F$  = " " "  $R^0E$  into  $R^0F$   
 (7)  $E$  with  $R^0F$  = " " "  $R^0E$  into  $F$   
 (8)  $R^0E$  with  $F$  = " " "  $E$  into  $R^0F$

where (5) into (6) into (7) into (8) = the enumerative zero.

From any one of the above eight formulae we can read off any other. Where any one mode of constructing an enumeration is equivalent in the above sense of = to a certain other mode of constructing an enumeration, it is obvious that the equivalence will imply equality of number, although the reverse does not hold: that is, we may have equality of number for two enumerations while the items comprised in them are not necessarily the same.

§ 2. We have now to consider how a single enumeration may be taken as an item to be enumerated along with other enumerations so as to constitute an *enumeration of enumerations*, that is an enumeration comprising, as its items, units which are themselves enumerations. This conception of an enumeration *comprising* enumerations must not be confused with an enumeration *including* enumerations. Thus: if ' $F$  includes  $E$ ' then the items comprised in  $E$  are the same as some of the items comprised in  $F$ , and here  $E$  and  $F$  comprise the same types or kinds of items. But if ' $F$  comprises  $E$ ,' then the items comprised in  $F$  will be of a higher order or type than the items comprised in  $E$ . Using the term item (in the first instance) to stand for an entity of order

zero, i.e., one which is not itself an enumeration, an enumeration comprising such items will be of the first order, and an enumeration comprising enumerations of the first order will be of the second order; and so on. In passing from enumerations of the first order, viz., those which comprise mere items, to enumerations of the second order which comprise enumerations of items, we may symbolise the distinction by using square brackets. Thus an enumeration of the first order may be illustrated thus:  $[a, b, c, d, e, f, g, h, k, l, m, n]$  involving one size of bracket. Now, with these same twelve items we may illustrate several enumerations of the second order which will involve *two* sizes of brackets as follows:

$$[[a, b], [c, d, e, f, g], [h, k, l, m], [n]]$$

where the item-enumerations are exclusive and four in number.

Again:

$$[[a, b, c], [c, d, e, f], [a, f, h, k, l], [d, m, n]]$$

where the item-enumerations are not exclusive of one another and again are four in number.

Or again:

$$[a, b], [c, d, e], [b, f, h], [d, k], [e, m, n]$$

where some pairs of the item-enumerations are exclusive and others not, the total number being five.

In all these illustrations, a comma is used to separate the items to be enumerated in constituting an enumeration, and the square brackets are used where required to indicate what is to be taken as an item. Similarly,

using the same twelve items we may illustrate an enumeration of the third order; which will involve three sizes of brackets thus:

$$\left[ \left[ [a, b], [c, d, e] \right], \left[ [b, f, h], [d, k] \right], \left[ [e, m, n] \right] \right],$$

where there are three items which are enumerations of the second order, of which the first two comprises as items two first order enumerations, while the third comprises only one first order enumeration. Comparing our illustration of a first-order enumeration with the first illustration of a second-order enumeration, we must note that the item  $n$  which is of zero order is to be distinguished from the item  $[n]$  which is of the first order, and comparing this last with our third order enumeration, we must distinguish the item  $[e, m, n]$  which is of the first order from  $\left[ [e, m, n] \right]$  which is of the second order. The distinction between  $n$  and  $[n]$  is, therefore, that the former is to count as one along with other items in constituting an enumeration of the *first* order, while  $[n]$  is to count as one along with other items in constituting an enumeration of the *second* order. Similarly, the distinction between  $[e, m, n]$  and  $\left[ [e, m, n] \right]$ , is that the former is to count as one along with other items in constituting an enumeration of the *second* order, while the latter is to count as one along with other items in constituting an enumeration of the *third* order. On precisely similar grounds we must distinguish between, say,  $\left[ [a, b], [c, d, e] \right]$  and  $\left[ \left[ [a, b], [c, d, e] \right] \right]$ ; for the former represents an enumeration of the second order

comprising *two* items which are enumerations of the first order, while the latter represents an enumeration of the third order comprising *one* item of the second order. In the above treatment we have in effect defined the notions 'item' and 'enumeration,' not as having absolute significance, but as having relative significance, in the sense that the two notions are indicated by the relative term 'comprising' and its correlative 'comprised in.' In other words our proper topic has been the development of the kind of relation expressed by the verb *comprise*.

Further to illustrate the principle that the operators *into* and *with* yield an enumeration of the same order as the enumerations operated upon, we will apply these operators to enumerations of the second order. When abbreviating the expression for an enumeration by substituting a single letter  $E$  or  $F$ , we shall use as indices 1, 2, 3, ... to indicate the different *orders* to which any enumeration may belong. Thus:

Let  $E^2$  stand for  $\left[ [a, b, c], [c, d], [a, e, f, g], [a, h, k, l] \right]$

and  $F^2$  stand for  $\left[ [a, b, c], [c, d, e], [a, e, f, g], [b, k, l, m] \right]$

then the operation ' $E^2$  into  $F^2$ ' yields  $\left[ [a, b, c], [a, e, f, g] \right]$

and the operation ' $E^2$  with  $F^2$ ' yields

$$\left[ [a, b, c], [c, d], [c, d, e], [a, e, f, g], [a, h, k, l], [b, k, l, m] \right]$$

Thus the operation  $E^2$  into  $F^2$  yields  $G^2$  and the operation  $E^2$  with  $F^2$  yields  $H^2$ , where it may be seen that  $G^2$  and  $H^2$  stand for enumerations of the second order. We shall also require a symbol for the result of using the operator *into*, where the enumerations are exclusive

of one another. According as the enumeration yielded in this case is of the first, second, or third order, it will be symbolised by  $[o]$ , or  $[[o]]$  or  $[[[o]]]$ . Thus:

The operation  $[a, b, c]$  into  $[d, e]$  yields  $[o]$ ; and the operation  $[[a, b, c], [c, d]]$  into  $[[a, d, e], [c, f]]$  yields  $[[o]]$ . Thus amongst all possible enumerations of the first order we must include  $[o]$ ; and amongst all those of the second order we must include  $[[o]]$ . An enumeration characterised as being  $o$  may be called an empty enumeration, the symbol  $o$  having in every case the same meaning: the single, double, etc., bracket *adds* a further character to the character zero. The symbol  $o$  is obviously selected to indicate that the number of items in an empty enumeration is zero. As regards items of zero order none can be called empty; and therefore none can be symbolised as  $o$ .

§ 3. At this point we must explain more precisely the distinction between being comprised in and being included in. Thus, each of the three items  $a, b, c$  is comprised in the enumeration  $[a, b, c]$ , and no others. The relation of comprising thus always correlates an item or an enumeration of a certain order with an enumeration of the *next higher* order. On the other hand  $[a]$  is included in  $[a, b, c]$ , thus showing that the relation of inclusion is a relation between enumerations of the *same* order.

The above account suggests the elementary problem, how many enumerations are included in the enumeration  $[a, b, c]$  (taken to be an enumeration of the first order).

Since  $[o]$  being of the first order is included in *every* first order enumeration, the following first order enumerations will exhaust all those that are included in  $[a, b, c]$ , namely:

First, that which comprises no item:  $[o]$ :

Secondly, those which comprise one item:

$[a]$  and  $[b]$  and  $[c]$ :

Thirdly, those which comprise two items:

$[a, b]$  and  $[a, c]$  and  $[b, c]$ :

Fourthly, that which comprises three items:  $[a, b, c]$ . Thus, within the enumeration  $[a, b, c]$  the number of distinct enumerations included is 8; for, in selecting any enumeration which shall be included in  $[a, b, c]$ , we have, with respect to each of these three items, the *two* alternatives of admitting or omitting it. Hence the number of our choices is  $2 \times 2 \times 2$ . Similarly in any enumeration comprising (say)  $n$  items,  $2^n$  enumerations will be included. Thus, we may write down an enumeration of the second order which shall comprise all the enumerations of the first order included in  $[a, b, c]$ .

Thus:  $[[o], [a], [b], [c], [a, b], [a, c], [b, c], [a, b, c]]$ . In general terms then: The enumeration of the second order, that shall comprise as its items all the enumerations of the first order included in a *given* enumeration of the first order comprising  $n$  items, will comprise  $2^n$  items of the first order.

§ 4. Having treated of enumerations we may now consider the relation between an enumeration and a class. Whether a class may or may not be considered as an enumeration of a special kind, it will be agreed that there is involved in the notion of a class an element

entirely absent from that of a mere enumeration. In the language of Mill, the denotation of a class may be said to be determined by connotation; i.e. by a certain conjunction of adjectives. But here it is of the utmost importance to note that, on the one hand, the substantial items constituting the denotation are united merely by the *enumerative* 'and'; but, on the other hand, the adjectival items constituting the connotation are united by the *conjunctive* 'and'.<sup>1</sup> In fact, what is common to every logician's employment of the term *class* is that its limits are determined—not merely, if at all, by a mere enumeration of items—but essentially by the character or conjunction of characters that can be truly predicated of this and of that item that is to be comprised. The distinction between an enumeration and a class is closely connected with that between an extensional and an intensional point of view, so that logicians have contrasted the extension of a class with its intension or the intensional conception of a class with its extensional conception. Phrases of this kind have in fact been introduced in various parts of this work; but a more direct way of attacking our problem would be to speak—not of the extension and the intension of a class—but of the extension of an intension and of the intension of an extension, where the preposition *of* requires to be logically defined. More explicitly I propose to speak of an intension as determining a certain extension, or conversely of an extension as being determined by a certain intension. Thus the relation *determining* and its correlative *determined-by* will indicate the required connection and distinction. We shall not generally

<sup>1</sup> See Chapter III.

speak of an extension determining an intension or of an intension being determined-by an extension, but the relation of determining will be always from the intension to the extension. In short, it is this direction of determination which justifies Mill's use of the term *connotation*, and when we are conceiving the converse case of an extension determining an intension, then we may adopt for the intension in this case the convenient term *comprehension* as introduced by Dr Keynes. The word *determining* as used above is of course elliptical. In speaking of a given intension or conjunction of adjectives as determining an extension, what of course is always understood is that this or that item is or is not to be comprised in the extension according as it is or is not characterised by the given conjunction of adjectives. Now it will be found that the larger and more familiar part of logical theory is actually concerned—not with the notion of extension—but solely with that of intension, and that it is only when arithmetical predicates come into consideration that the notion of extension seems to be required. Thus, taking the proposition: 'Everything having the character *m* has the character *p*,' we may, for any English letter standing illustratively for an adjective, introduce the corresponding Greek letter in a purely *symbolic* sense to stand for the class determined by that adjective. Thus the above distributively expressed proposition may be rendered: 'the class  $\mu$  is included in the class  $\pi$ .' Again, if we conjoin with the above proposition: 'Everything having the character *p* has the character *m*,' we reach the form of proposition: 'The class  $\mu$  coincides with the class  $\pi$ .' Now the relation *coincides* is analogous to the relation

of co-implication, in that both are transitive, symmetrical and reflexive; i.e., they have the properties of equivalence or identity. In this way we may speak of the identity of a class determined by one adjective with that determined by another (merely as expressing a symbolic or abbreviated formula) without implying that there is any real entity to be called an extension or a class to which the strict relation of identity could be applied. All this is assumed in the next chapter, where we shall represent the force of propositions by means of closed figures. In spite then of the prominent employment of the word *class* both in the treatment of propositions and still more in that of the principles of syllogism, it may be maintained that there is no real reference in thought to the class as an extension, but only a figurative or metaphorical application of the word which serves to bring out certain analogies between such notions as inclusion, exclusion, and exhaustion which apply primarily to parts and wholes and are transferred as relations between propositions and their constituent elements. Some logicians have even gone so far as to say that the spatial relations amongst plane closed figures represent the actual mode of thought by means of which we are able to comprehend logical relations. I, however, reject this extreme point of view, but suggest that the mere fact that we are able to represent logical relations by analogy with relations amongst spatial figures almost justifies our maintaining that the idea of an extension determined by an intension is a logically valid concept. The full significance of such a scheme as Euler's diagrams for representing class-relationships has, in my view, been inadequately

recognised. It should be pointed out that the boundary line of a closed figure may be taken as the proper analogue of the intension, while the area within that boundary is the proper analogue of the extension. This suggestion brings out the following analogies: firstly, that it is *i.* intension which determines extension in the same way as a boundary line determines the enclosed area and separates this area from the remaining area outside; secondly, that we can apprehend in thought the full *ii.* determining intension in the same way as we can optically grasp the single boundary in its entirety; and thirdly, that in general we cannot in thought enumerate *iii.* all the items which are to be comprised in the extension, just as we cannot exhaustively present to the eye the several and innumerable points within the given enclosed area. On the other hand, though the several points cannot be exhaustively presented to the eye and yet *β.* the area presents itself ocularly as a unitary whole, similarly it would seem that though we cannot exhaustively enumerate in thought the members of a class yet we can conceive the class or rather the extension as a unitary whole. Again we may make within the area actual dots of a finite number which thus constitute *γ.* a literal (though of course not exhaustive) enumeration, and thus the force of the diagram, as providing analogues to logical relations, is still further brought out, in that we may think one by one of the objects which we have selected—not arbitrarily—but on the ground that each of them is actually characterised by the adjectives which determine the class. Whether this analogy between a psychical image or perception of an area and the logical conception of a class, justifies our regarding the latter as



a genuine concept, is a debatable psychological problem. Dismissing this problem we must return to the strictly logical question whether a class is a genuine entity.

§ 5. If we provisionally allow a class comprised of individual existents to be an existent, then as an existent it is of a different order from any individual comprised in it. Similarly a class comprising adjectives—if it is to be called an adjective—must be an adjective of a different order from any adjective comprised in it. Similarly for a class of propositions. Therefore the question whether a class is a genuine entity admits at any rate that it is not of the same order of being as any item which it comprises. Whether this or that is a genuine entity can only be answered when we have provided a test of genuineness. The only general test which I can conceive of is as to whether the entity intended to be meant (in using such a word as *class*) can serve as subject of which some predicate can be truly asserted<sup>1</sup>. Thus, as an illustration of the *general* question, we may ask whether a proposition is a genuine entity, and taking the proposition *matter exists*, the reply would be in the affirmative, inasmuch as we can make the assertion 'that *matter exists* was rejected by Berkeley.'

Similarly with regard to the genuineness of a 'class' which is the topic under consideration. Taking for example the class *apostles*, we may assert 'that this class numbers twelve.' Inasmuch as this statement will be admitted to be true, the only relevant question that could arise would be as to whether the number *twelve* is predicated—*not* of the kind of entity called a class—but rather of its determining adjective. To this it may

<sup>1</sup> Cf. the treatment of 'S is' in Chapter V.

be replied that since different intensions may determine the same class, the number predicated cannot be said to be predicated of one of these determining intensions rather than of any other. In other words, we may say that the adjective *twelve* is of such a kind that, taking any two co-implicative determining intensions, if it can be truly predicated of one it can be also truly predicated of the other. Since then it is indifferent of which of these several co-implicatives the adjective *twelve* is to be predicated, it seems to follow that it is not predicated of any of them whatever. Let us take *p* and *q* to stand for two co-implicative intensions. This means more precisely that, if anything is characterised as being *p*, it will be characterised also as being *q*; and conversely, if anything is characterised as being *q*, it will be also characterised as being *p*. It is then clear that the relation called co-implication is both symmetrical and transitive. Let us then assume the question at issue, namely that there is such an entity as a class. Then our conception of a class involves the universal statement that any given intension determines one and only one class. In this way the relation of co-implication subsisting between *p* and *q*, may be resolved into the statement that *p* determines a certain unique entity which is the same as that which is determined by *q*: that entity being what we have taken to be a class. Here the relation *determining*, which relates an intension to its extension, is what is called a many-one relation, because there may be many different intensions which determine a single extension. Similarly, the relation *determined-by*, which relates an extension to its intension, is what is called a one-many relation, because there is only one extension which is determined by many different intensions.



§ 6. We may generalise the above by taking, as a typical illustration, the relation of a *man* to the *country* which he inhabits. Since there is only one country which a man inhabits the relation *inhabits* is many-one, and its converse *inhabited-by* is one-many. Now let us combine these two correlatives in the following form: '*A* inhabits *the* country that is inhabited by *B*.' The relation thus constructed of *A* to *B* (which may be called the relation *compatriot-of*) is obviously symmetrical and transitive. It is symmetrical because there is only one country which *A* can inhabit and one country only which *B* can inhabit, so that to say that *A* and *B* inhabit the same country exhibits a symmetrical relation, since the terms *A* and *B* may be interchanged. Again, the relation is transitive since if *A* and *B* inhabit the same country, and *B* and *C* also inhabit the same country, it follows, since no one inhabits more than one country, that *A* and *C* inhabit the same country. The *identity* of the country inhabited by this, that and another man is a symmetrical and transitive relation, and it is upon these properties of identity that the symmetry and transitivity of the relation *compatriot-of* depends. In fact, to speak of *the* country inhabited by *A* would not be a legitimate expression unless there was *one and only one* country which a man could be said to inhabit. Thus it will be seen that, when we combine in this sort of way any many-one relation with its correlative (which is necessarily one-many), then we have constructed a relation which has the two properties transitive and symmetrical. But the reverse of this does not obviously hold—that is, given a transitive and symmetrical relation it does not obviously follow that this relation can be resolved into a combination of a certain many-one

relation with its correlative. If, however, this converse proposition may be taken as axiomatic, it would follow, from the two properties—symmetrical and transitive—which hold of co-implication between two intensions  $p$  and  $q$ , that there is a certain thing which  $p$  determines and which is determined by  $q$ . To this kind of entity we apply the name *class*. It is therefore only by *assuming* the theorem that any given relation that is symmetrical and transitive can be resolved as above in terms of a many-one relation and its converse that we can *prove* that the notion of a class represents a genuine entity. But, in making use of the analogy between (a) 'the country inhabited by any of the men that are compatriots of one another' and (b) 'the class determined by any of the intensions that are co-implicants of one another,' it must be further pointed out that, just as the 'country' under consideration is not the same as the compatriots taken in their totality, so the 'class' under consideration is not the same as the co-implicants taken in their totality. Since, then, the 'compatriots' or the 'co-implicants' taken in their totality would constitute a *class*, the attempt to *prove* the above theorem would entail a *petitio principii*, when applied to the question of the genuineness of the notion 'class.' Merely from the symmetry and transitivity of the relation ( $\hat{s}$ ) compatriot, we cannot prove that there is a certain many-one relation ( $\hat{r}$ ) inhabit, or (an entity  $x$ ) viz. a certain country *distinct from the class of compatriots all of whom inhabit ( $\hat{r}$ ) that country ( $x$ )*. Our certainty that 'when  $a$  is  $\hat{s}$  to  $b$ , then  $a$  is  $\hat{r}$  to  $x$  and  $x$  is  $\hat{r}$  to  $b$  for some  $x$ ' is due to the fact that, in order to construct any  $\hat{s}$ , we must first have been given  $\hat{r}$ .

## X CHAPTER IX

THE GENERAL PROPOSITION AND ITS  
IMMEDIATE IMPLICATIONS<sup>1</sup>

§ 1. WE may define a general proposition as one in which the subject is constructed by prefixing an applicative to a general name. According to this definition, the only kind of proposition which is not general would be that in which the subject is expressed by a proper name; and the general proposition would include two forms of singular proposition: namely, where the general subject-term is prefixed by 'a certain' or by 'the.'

We must first point out that there are what may be called *pure* general propositions, where the general term is represented by the word 'thing' in its absolute universality: for example 'Everything is finite,' 'some things are extended.' In these propositions the subject-term has merely substantival without any adjectival significance. But ordinary propositions in every-day use apply to subjects adjectivally restricted; in other words, there is an adjectival significance in the subject-term as well as in the predicate term.

§ 2. Using the capital letters *P* and *Q* for general or class terms and the corresponding small letters *p* and *q* to represent their adjectival significance, then the correct

<sup>1</sup> This Chapter should be read in close connection with Chapter III.

expression for the affirmatives, universal and particular, would be:

Every *P* is *q*,

Some *P* is *q*,

which brings out the substantival significance in the subject-term and the adjectival significance of the predicate term. Now these forms are at once seen to be equivalent respectively to

*A.* Everything that is *p* is *q*,

*I.* Something that is *p* is *q*,<sup>1</sup>

where the adjective *p* occurs as predicate in the subordinate clause, and the introduction of the word 'thing' indicates an ultimate reference to the absolutely general substantive. The *negative* general categoricals may first be expressed in the forms:

Everything that is *p* is-not *q*,

Something that is *p* is-not *q*,

where, in attaching the negative to the copula, it is to be understood that the adjective *q* is *denied*, in the first case, of everything that is *p*; and, in the second case, of something that is *p*. But these negatives are expressed less ambiguously as the contradictories of *I* and *A* respectively, thus:

*E.* Nothing that is *p* is *q*,

*O.* Not-everything that is *p* is *q*,

<sup>1</sup> The subject-terms in these two readings are often contrasted as being the one in denotation and the other in connotation. This disregards the fact that the meaning of the term *P* contains a connotative as well as a denotative factor, while the phrase 'thing that is *p*' contains a denotative as well as a connotative factor. Hence the two readings are not properly to be contrasted.

where the negative is prefixed to the propositions as a whole, and must not be falsely supposed to qualify the *subject-term*.

By expressing the propositions  $A, I, E, O$ , in the above forms the true logical nature both of obversion and of conversion can be explained. Thus the negative that is introduced or omitted by the process of obversion, is to be attached to the adjectival factor alone in the predicate; and hence the four propositions:

|                |                             |
|----------------|-----------------------------|
| $A.$ Every     | } -thing that is $p$ is $q$ |
| $E.$ No        |                             |
| $I.$ Some      |                             |
| $O.$ Not-every |                             |

become (respectively) by obversion

|                  |                                  |
|------------------|----------------------------------|
| $E_b.$ No        | } -thing that is $p$ is non- $q$ |
| $A_b.$ Every     |                                  |
| $O_b.$ Not-every |                                  |
| $I_b.$ Some      |                                  |

where  $E_b, A_b, O_b, I_b$ , are of the forms  $E, A, O, I$ , but having in each case non- $q$  for predicate in place of  $q$ . Here the suffix  $b$  indicates that the proposition is obtained by obversion. Applying again the process of obversion to  $E_b, A_b, O_b, I_b$ , we obtain

|                     |                                        |
|---------------------|----------------------------------------|
| $A_{bb}.$ Every     | } -thing that is $p$ is non-non- $q$ . |
| $E_{bb}.$ No        |                                        |
| $I_{bb}.$ Some      |                                        |
| $O_{bb}.$ Not-every |                                        |

Now, by the principle of double negation, non-non- $q$

is equivalent to  $q$ . Hence  $A_{bb}, E_{bb}, I_{bb}, O_{bb}$ , are respectively equivalent to  $A, E, I, O$ —showing that the propositions obtained by obversion are equipollent, i.e. formally coimplicant.

The process of simple conversion may be exhibited by interpolating intermediate steps, where first the adjectival factor in the subject is removed to the predicate; then, by the commutative law for conjunctives, the adjectives are transposed; and finally, the first adjectival factor in the predicate is removed back to the subject:

$E.$  Nothing that is  $p$  is  $q$   
 = Nothing is  $p$  and  $q$   
 = Nothing is  $q$  and  $p$   
 = Nothing that is  $q$  is  $p$ .

$I.$  Something that is  $p$  is  $q$   
 = Something is  $p$  and  $q$   
 = Something is  $q$  and  $p$   
 = Something that is  $q$  is  $p$ .

The forms  $A$  and  $O$ , not being directly convertible, must first be obverted and then converted, giving the contrapositive, i.e. the converted obverse, thus:

$A.$  Everything that is  $p$  is  $q$   
 = Nothing that is  $p$  is non- $q$   
 = Nothing that is non- $q$  is  $p$ .

$O.$  Not everything that is  $p$  is  $q$   
 = Something that is  $p$  is non- $q$   
 = Something that is non- $q$  is  $p$ .

We have so far taken  $E$ , 'No  $P$  is  $q$ ' to be equivalent to 'Nothing is  $pq$ '; and  $I$ , 'Some  $P$  is  $q$ ' to be equivalent to 'Something is  $pq$ .' In other words we

have reformulated the  $E$  and  $I$  propositions, in which the subject-term is restricted by the adjective  $p$ , by using as subject-term the unrestricted reference expressed by the word 'thing.' In the same way,  $A$ , 'Every  $P$  is  $q$ ' and  $O$ , 'Not-every  $P$  is  $q$ ' may be reformulated thus:  $A$ , 'Everything is  $\bar{p}$  or  $q$ '; and  $O$ , 'Not-everything is  $\bar{p}$  or  $q$ '; since ' $\bar{p}$  or  $q$ ' is equivalent to ' $q$  if  $p$ .' In these reformulations, the form of the proposition as universal or particular and as negative or affirmative is unaltered: the  $E$ -proposition is expressed as an  $E$ -proposition, the  $I$  as an  $I$ , the  $A$  as an  $A$ , the  $O$  as an  $O$ . We have, in short, merely reduced all the propositions to the same common denominator (so to speak) by using the narrowest reference for our subject-term which will be sufficiently wide to include all the subject-terms that may occur in any given connected system of statements. This kind of transformation has been called 'existential,' but since the term 'existential' has been so persistently misunderstood, it will be preferable to speak of 'instantial' instead of 'existential' formulation. In this mode of formulation, a further question arises, namely that of *interpretation*: in particular, as to whether the proposition—given to be reformulated—is to be understood to include (implicitly or explicitly) the statement that there are instances characterised by  $p$ , where  $p$  is the adjective connoted by the subject-term: i.e. to include the affirmative statement 'Something is  $p$ .' Here it is to be observed that the instantial statement 'Something is  $p$ ' is implicitly contained in 'Something is  $pq$ ,' but *not* in 'Nothing is  $pq$ .'

§3. We will distinguish the proposition which *contains* the instantial affirmation of its subject-adjective from

the (otherwise) same proposition which does *not* contain this affirmation by using the suffix  $f$  for the former, and  $n$  for the latter. In order to change (say)  $T_n$  into  $T_f$ , where  $T$  stands for any proposition having an adjectival factor (say  $p$ ) in its subject-term, we must *add to*  $T_n$  the statement 'Something is  $p$ ,' i.e. we must *conjunctively* combine with  $T_n$  the instantial *affirmation*. And, in order to change (say)  $T_f$  into  $T_n$ , we must *subtract from*  $T_f$  the statement 'Something is  $p$ ,' i.e. we must *alternatively* combine with  $T_f$  the instantial *denial*. Thus: given  $T_n$ ,  $T_f$  will be expressed ' $T_n$  and Something is  $p$ '; given  $T_f$ ,  $T_n$  will be expressed ' $T_f$  or Nothing is  $p$ .'

Applying these expressions to the forms  $E_n, I_f, A_n, O_f$  we have:

|       |       |                                                |
|-------|-------|------------------------------------------------|
| $E_n$ | means | 'Nothing is $pq$ .'                            |
| $I_f$ | "     | 'Something is $pq$ .'                          |
| $A_n$ | "     | 'Nothing is $p\bar{q}$ .'                      |
| $O_f$ | "     | 'Something is $p\bar{q}$ .'                    |
| $E_f$ | means | 'Nothing is $pq$ and Something is $p$ .'       |
| $I_n$ | "     | 'Something is $pq$ or Nothing is $p$ .'        |
| $A_f$ | "     | 'Nothing is $p\bar{q}$ and Something is $p$ .' |
| $O_n$ | "     | 'Something is $p\bar{q}$ or Nothing is $p$ .'  |

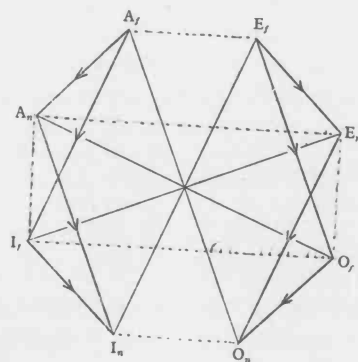
From the meanings of the symbols, as thus shown, the following rules will be evident:

(a) for obtaining the co-opponent: interchange  $f$  with  $n$  and  $A$  with  $O$ ; or  $f$  with  $n$  and  $E$  with  $I$  (without any other change);

(b) for obtaining a sub-implicant; change either  $f$  into  $n$ , or  $A$  into  $I$ , or  $E$  into  $O$  (without any other change).

Since super-implication is the reverse of sub-implication, rule (b) reversed shows how to obtain a super-

implicant. Moreover, since a sub-opponent is a sub-implicant of the co-opponent and a super-opponent is a super-implicant of the co-opponent, rules (a) and (b) combined show how to obtain a sub-opponent or super-opponent. Lastly, since a sub-implicant of a sub-implicant is a sub-implicant, and a super-implicant of a super-implicant is a super-implicant, all the several sub-implicants, super-implicants, sub-opponents or super-opponents can be found. In the following mnemonic diagram, an arrow is used to point from a super-implicant to a sub-implicant, and co-opponents are placed diagonally opposite. The diagram is required in place of the ordinary square of implication and opposition, because of the distinction introduced between two possible interpretations of  $A$ ,  $E$ ,  $I$ , or  $O$ .



Here,

$E_f, E_n, O_f$  are super-implicants of  $O_n$ , and are  $\therefore$  super-opponents of  $A_f$ ,  
 $A_f, A_n, I_f$  are super-implicants of  $I_n$ , and are  $\therefore$  super-opponents of  $E_f$ ,  
 $I_n, I_f, A_n$  are sub-implicants of  $A_f$ , and are  $\therefore$  sub-opponents of  $O_n$ ,  
 $O_n, O_f, E_n$  are sub-implicants of  $E_f$ , and are  $\therefore$  sub-opponents of  $I_n$ .

But, taking the laterally outstanding rectangle  $A_n, E_n, O_f, I_f$ , it must be observed that *no* relation of implication or opposition holds of  $A_n$  to  $E_n$ , of  $E_n$  to  $O_f$ , of  $O_f$  to  $I_f$ , or of  $I_f$  to  $A_n$ ; i.e. the sides of this rectangle exhibit the relation of *independence*.

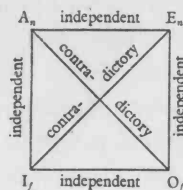
The general nature of these results is that where any proposition is interpreted as having less determinate significance, it will be a super-implicant or super-opponent of fewer propositions and a sub-implicant or sub-opponent of a larger number. Thus  $A_f$  is super-implicant to  $A_n$  and  $I_f$  and  $I_n$ , but  $A_n$  is super-implicant only to  $I_n$ ; and  $A_f$  is super-opponent to  $E_f$  and  $E_n$  and  $O_f$ , but  $A_n$  (being sub-opponent to  $O_n$ ) is super-opponent to none of the propositions in the octagon. Conversely where any proposition is interpreted as having more determinate significance, it will be sub-implicant or sub-opponent to fewer propositions and super-implicant or super-opponent to a larger number.

Similar modifications of the traditional scheme are required for inferences involving conversion. It will be found that equipollent conversion holds for  $E_n$  and  $I_f$  but not for  $E_f$  and  $I_n$ ; and that sub-altern conversion holds in passing to  $I$  from  $A_f$  but not from  $A_n$ ; and in passing from  $E$  to  $O_n$  but not to  $O_f$ .

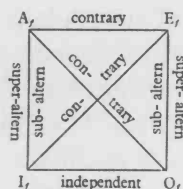
Since each of the propositions  $A, E, I, O$  can be interpreted in two ways, there are several possible schemes of interpretation of the four together, in accordance with which a square can be extracted from the above octagon. Of these combinations, the following are the most reasonable.

(1)  $A_n E_n I_f O_f$ .

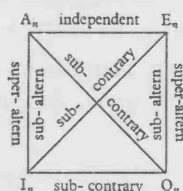
Here the universals are interpreted as containing no instancial affirmation, while the particulars implicitly contain instancial affirmation of the subject-term. This is the simplest interpretation, since each proposition is expressed as *uncompounded*, the universals being merely instancial denials, and the particulars, merely instancial affirmations.

(2)  $A_f E_f I_f O_f$ .

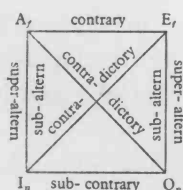
Here all the four propositions are interpreted as containing instancial affirmation of the subject-term; so that the *universals* have to be expressed in a compound form.

(3)  $A_n E_n I_n O_n$ .

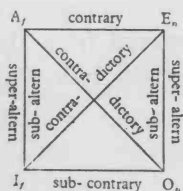
Here all the four propositions are interpreted as containing no instancial affirmation, so that the *particulars* have to be expressed in a compound form.

(4)  $A_f E_f I_n O_n$ .

This is the reverse of the first interpretation, each proposition being expressed as a compound: the universals containing instancial affirmation of the subject-term, and the particulars containing no instancial affirmation.

(5)  $A_f E_n I_f O_n$ .

Here the affirmatives contain instancial affirmation of the subject-term, and the negatives contain no instancial affirmation.



§ 4. These five selected schemes must be compared with the traditional doctrine on the relations of  $A, E, I, O$ . Special explanation is required to justify the application of the terms contrary, contradictory, etc., in the traditional scheme, where all the four general propositions are to be understood to assume that there are instances of the subject-term. This interpretation differs from interpretation (2) given above, where each proposition is interpreted as containing this affirmation; for, in case there were no instances of the subject-term, either of the four forms of proposition would, on the traditional scheme be *meaningless*, whereas on interpretation (2) they would be *false*. Thus  $I_f$  and  $O_f$  would both be false, i.e.  $A_n$  and  $E_n$  would both be true, supposing there to be no instances of the subject-term; whereas the traditional scheme, precluding this possibility from the outset, asserts that  $I$  and  $O$  cannot both be false, i.e. that  $A$  and  $E$  cannot both be true. This presupposition of traditional logic is concealed from the ordinary reader by the universal employment of Euler's diagrams, in which the subject-class is indicated by an actual circle, so that the limiting case, where the class vanishes, is never represented. Furthermore, the conversions authorised on the traditional scheme derive their validity from the assumption that there are instances not only of  $p$ , but also of  $q$  and of non- $p$  and of non- $q$ , where  $p$  and  $q$  are the adjectival factors in the subject and predicate terms. These assumptions again are tacitly involved in Euler's diagrams, where the circles for  $P$  and  $Q$  are not allowed to vanish or to exhaust the universe.

§ 5. In what follows we shall adopt the traditional view that there are instances of  $p$ , of non- $p$ , of  $q$ , and of



non- $q$ ; and on this assumption we proceed to consider all the formal relations amongst the propositions involving  $p$  or non- $p$  with  $q$  or non- $q$ . The symbols  $A', E', I', O'$  will stand respectively for the propositions  $A, E, I, O$  when modified by negating both the subject and the predicate adjective. Thus, using the following abbreviative substitutions, viz., 'All' for 'every,' ' $p$ ' for 'thing that is  $p$ ,' and ' $\bar{p}$ ' for non- $p$ , we have

|                          |                                       |
|--------------------------|---------------------------------------|
| $A =$ All $p$ is $q$     | $A' =$ All $\bar{p}$ is $\bar{q}$     |
| $E =$ No $p$ is $q$      | $E' =$ No $\bar{p}$ is $\bar{q}$      |
| $I =$ Some $p$ is $q$    | $I' =$ Some $\bar{p}$ is $\bar{q}$    |
| $O =$ Not all $p$ is $q$ | $O' =$ Not all $\bar{p}$ is $\bar{q}$ |

giving a list of eight *distinct* (i.e. non-equipollent) general categoricals, as an extension of the usual four. Adding to this list the propositions whose subject and predicate terms are  $p\bar{q}, \bar{p}q, q\bar{p}, \bar{q}p, q\bar{p}, \bar{q}p$  we obtain in all 32 categoricals; of non-equipollents, however, there are only  $32 \div 4$ , since by means of obversion and simple conversion, each proposition may be expressed in four *equipollent* forms. This is shown in

TABLE I

|      | By Obversion<br>(i)            | By Conversion<br>(ii)       | By Obversion<br>(iii)       | By Conversion<br>(iv)          |
|------|--------------------------------|-----------------------------|-----------------------------|--------------------------------|
| $A$  | All $p$ is $q$                 | No $\bar{p}$ is $\bar{q}$   | No $\bar{q}$ is $\bar{p}$   | All $\bar{q}$ is $\bar{p}$     |
| $A'$ | All $\bar{p}$ is $\bar{q}$     | No $p$ is $q$               | No $q$ is $p$               | All $q$ is $p$                 |
| $E$  | All $p$ is $\bar{q}$           | No $\bar{p}$ is $q$         | No $q$ is $\bar{p}$         | All $q$ is $\bar{p}$           |
| $E'$ | All $\bar{p}$ is $q$           | No $p$ is $\bar{q}$         | No $\bar{q}$ is $p$         | All $\bar{q}$ is $p$           |
| $O$  | Not all $p$ is $q$             | Some $\bar{p}$ is $\bar{q}$ | Some $\bar{q}$ is $\bar{p}$ | Not all $\bar{q}$ is $\bar{p}$ |
| $O'$ | Not all $\bar{p}$ is $\bar{q}$ | Some $p$ is $q$             | Some $q$ is $p$             | Not all $q$ is $p$             |
| $I$  | Not all $p$ is $\bar{q}$       | Some $\bar{p}$ is $q$       | Some $q$ is $\bar{p}$       | Not all $q$ is $\bar{p}$       |
| $I'$ | Not all $\bar{p}$ is $q$       | Some $p$ is $\bar{q}$       | Some $\bar{q}$ is $p$       | Not all $\bar{q}$ is $p$       |

In this table column (i) gives propositions of the form  $A$  or  $O$  (which admit of simple contraposition); the second column is derived by obversion, giving propositions of the form  $E$  or  $I$  (which admit of simple conversion); the third is next derived by (simple) conversion, giving again propositions of the form  $E$  or  $I$ ; and the fourth is derived again by obversion, giving propositions of the form  $A$  or  $O$ . The processes of obversion and simple conversion being reciprocal, give equipollents, i.e. formal co-implicants. The relation of contradiction (i.e. formal co-opposition) is seen in the first instance from the predesignations 'All' versus 'Not all' and 'Some' versus 'No,' and, on account of the equipollences tabulated, each of the four universal propositions in any the same row is related as contradictory to each of those in the ordinally corresponding row of particulars: i.e.  $A$  to  $O$ ,  $A'$  to  $O'$ ,  $E$  to  $I$ ,  $E'$  to  $I'$ .

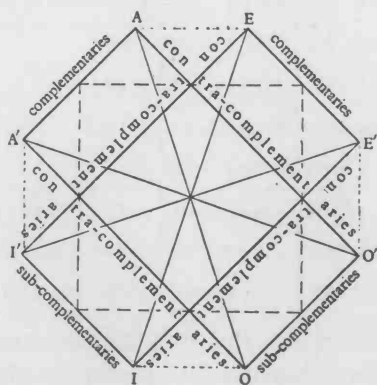
TABLE II

|                         | $A$                            | $E$                            | $I$                      | $O$                         |
|-------------------------|--------------------------------|--------------------------------|--------------------------|-----------------------------|
| Verse ...               | All $p$ is $q$                 | No $p$ is $q$                  | Some $p$ is $q$          | Not all $p$ is $q$          |
| Obverse ...             | No $\bar{p}$ is $\bar{q}$      | All $\bar{p}$ is $\bar{q}$     | Not all $p$ is $\bar{q}$ | Some $\bar{p}$ is $\bar{q}$ |
| Contrapositive ...      | No $\bar{q}$ is $\bar{p}$      | Some $\bar{q}$ is $\bar{p}$    |                          | Some $\bar{q}$ is $p$       |
| Obverted Contrapositive | All $\bar{q}$ is $\bar{p}$     | Not all $\bar{q}$ is $\bar{p}$ |                          | Not all $\bar{q}$ is $p$    |
| Inverse ...             | Some $\bar{p}$ is $\bar{q}$    | Not all $\bar{p}$ is $\bar{q}$ |                          |                             |
| Obverted Inverse        | Not all $\bar{p}$ is $\bar{q}$ | Some $\bar{p}$ is $\bar{q}$    |                          |                             |
| Obverted Converse       | Not all $q$ is $\bar{p}$       | All $q$ is $\bar{p}$           | Not all $q$ is $p$       |                             |
| Converse ...            | Some $q$ is $\bar{p}$          | No $q$ is $\bar{p}$            | Some $q$ is $p$          |                             |
| Verse ...               | All $p$ is $q$                 | No $p$ is $q$                  | Some $p$ is $q$          | Not all $p$ is $q$          |

Table II gives all the implications amongst the given propositions that can be drawn by alternate ob-

version and conversion, beginning first with obversion, and next with conversion. The arrows in the table show the direction of the inference, and where there is no arrow there is no inference. This table includes the equipollents of Table I; and contains also the sub-altern conversions which are required for the form *A*. When a proposition of the form *O* is to be converted the process must stop. The same table could be written out for *A'*, *E'*, *I'*, *O'*. Having in this way given exhaustively the relations of equipollence, contradiction, sub- and super-implication and opposition, it remains to deal with the relation of *independence*. This will hold between the following pairs: *A* and *A'*, *E* and *E'*, *A* and *O'*, *A'* and *O*, *E* and *I'*, *E'* and *I*, for which we shall use the technical terms complementary for independence between universals, sub-complementary for independence between particulars, and contra-complementary for independence between a universal and a particular.

All these results are expressed in the following octagon of implication and opposition.



§ 6. The above account of the processes of sub-alternation, obversion and conversion of general categorical propositions is based upon the logical relations amongst compound propositions explained in Chapter III, by applying these latter without modification in the form of logical relations amongst adjectives. Now this correspondence may be further developed by bringing out the analogies between the universal and particular forms of categorical proposition on the one hand, and what was called in Part I Chapter III the necessary and possible forms of the compound proposition on the other hand. Thus, the form '*p* implies *q*,' where the relation asserted of the two component propositions is irrespective of their truth or falsity, is naturally contradicted in the form '*p* does not imply *q*,' where again the relation is asserted irrespective of the truth or falsity of these components. Analogously, taking *p* and *q* to be adjectives (instead of propositions) the categorical 'Everything that is *p* is *q*,' where a relation of *p* to *q* is asserted irrespective of any *given* thing being *p* or *q*, is naturally contradicted in the form 'Not everything that is *p* is *q*,' where again a relation is asserted of *p* to *q* irrespective of any *given* thing being *p* or *q*. Expressing the compound propositions in terms of possible or impossible, the proposition '*p* with not-*q* is impossible,' contradicts '*p* with not-*q* is possible,' these compounds being respectively analogous to the universal 'Nothing that is *p* is non-*q*' and the particular 'Something that is *p* is non-*q*.' Thus there is a literal equivalence in the relations subsisting amongst the 'necessary composites' and 'possible conjunctives' on the one hand, and those subsisting amongst the 'universal' and 'particular' cate-

goricals on the other hand. Some logicians indeed have demanded that Logic should *interpret* the universal and particular categoricals to stand respectively for necessary implication and possible conjunction; this view, however cannot be accepted. The analogy that properly holds demands equivalence—not in the forms of proposition themselves—but in the logical relations amongst them. The basis for these analogies is shown in the following fundamental forms, where  $p$  and  $q$  are to stand for adjectives:

| General<br>Categorical Form      |       | Adjectivally<br>Compound Form |
|----------------------------------|-------|-------------------------------|
| A. Every $p$ is $q$              | = $p$ | implies $q$                   |
| A'. Every $q$ is $p$             | = $p$ | is implied by $q$             |
| E. No $p$ is $q$                 | = $p$ | is co-disjunct to $q$         |
| E'. Everything is $p$ or $q$     | = $p$ | is co-alternate to $q$        |
| O. Not every $p$ is $q$          | = $p$ | does not imply $q$            |
| O'. Not every $q$ is $p$         | = $p$ | is not implied by $q$         |
| I. Some $p$ is $q$               | = $p$ | is not co-disjunct to $q$     |
| I'. Not everything is $p$ or $q$ | = $p$ | is not co-alternate to $q$    |

The four relations and their contradictories here exhibited lead by combination to seven possible relations corresponding exactly to those shown in Part I Chapter III. Thus:

|                                                          |                                 |
|----------------------------------------------------------|---------------------------------|
| A and A'. Every $p$ is $q$ and Every $q$ is $p$          | = $p$ is co-implicant to $q$    |
| A and O'. Every $p$ is $q$ and Not every $q$ is $p$      | = $p$ is super-implicant to $q$ |
| A' and O. Every $q$ is $p$ and Not every $p$ is $q$      | = $p$ is sub-implicant to $q$   |
| O and O' and I and I'.                                   | = $p$ is independent of $q$     |
| E' and I. Everything is $p$ or $q$ and Some $p$ is $q$   | = $p$ is sub-opponent to $q$    |
| E and I'. No $p$ is $q$ and Not everything is $p$ or $q$ | = $p$ is super-opponent to $q$  |
| E and E'. No $p$ is $q$ and Everything is $p$ or $q$     | = $p$ is co-opponent to $q$     |

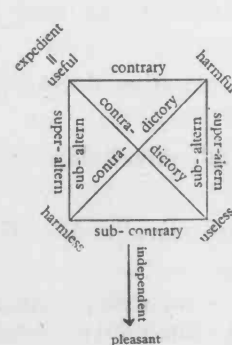
Thus the same seven relations in which propositions may stand to one another hold of the relations in which

adjectives may stand when entering into the subject and predicate of universal and particular propositions.

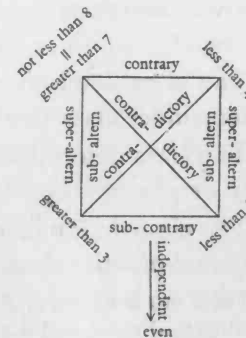
If  $p$  is formally independent of  $q$ , then all the relations tabulated above are material, but if  $p$  is formally related to  $q$ , then it must be related in one or other of the six ways, which remain when the relation of independence is excluded. For example, taking the five adjectives *useful*, *harmful*, *useless*, *harmless*, and *expedient*, which are formally related, and adding *pleasant* which is formally unrelated, we find that:

- (i) expedient is equipollent to useful,
- (ii) useful is super-altern to harmless,
- (iii) useless is sub-altern to harmful,
- (iv) [useful is independent of pleasant],
- (v) useless is sub-contrary to harmless,
- (vi) useful is contrary to harmful,
- (vii) useful is contradictory to useless.

The same relations hold for the five numerical adjectives—greater than 7, less than 4, less than 8, less than 3, not less than 8, to which we may add 'even' in place of 'pleasant.' Thus:



J. L.



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§ 7. Yet another way of representing the categorical proposition is in terms—not of the adjectives  $p$  and  $q$ —but of the substantive classes  $P$  and  $Q$  which these adjectives determine. Any class must be conceived in extension as a part of the universe of substantives, where the universe is sufficiently widely extended to include all the classes which occur in any set of interconnected propositions. The absolutely widest substantive universe is that which we have represented by the word 'thing'; and corresponding to any more restricted universe, the same word 'thing' can be used with a correspondingly understood restriction.

The following three technical terms may now be introduced: (a) the part of the universe which remains when any given class is subtracted will be denominated the *remainder* (or *co-remainder*) to the class; (b) the smallest class that *includes* both of two given classes  $P$  and  $Q$  will be denominated ' $P$  with  $Q$ '; and (c) the largest class that *is included in* both of two given classes  $P$  and  $Q$  will be denominated ' $P$  into  $Q$ .' With the help of these three class functions, the following fundamental relations between the class functions and the adjectival functions which determine them may be expressed:

- (1) The class determined by the negative not- $p$  = the remainder class to  $P$ .
- (2) The class determined by the adjectival alternation ' $p$  or  $q$ ' = the class ' $P$  with  $Q$ .'
- (3) The class determined by the adjectival conjunction ' $p$  and  $q$ ' = the class ' $P$  into  $Q$ .'

<sup>1</sup> These two functions of  $P$  and  $Q$  have many of the properties of the arithmetical L.C.M. and H.C.F. See also Chapter VIII.

Thus the notions 'not,' 'or,' 'and' which must be applied to predications (and here to adjectives) *correspond* respectively to the notions 'remainder,' 'with,' 'into,' which apply to substantive classes. Traditional logic has encouraged confusion between these two types of notion by employing the terms which are only proper for adjectival functions, for class functions also<sup>1</sup>. This usage, while it has the advantage of brevity and facilitates the logical transformations that the learner has to carry out, suffers from the serious objection of leading to confusion between the two types of function. Thus the notion of the remainder as a relation between classes is founded upon that of non-identity as a relation between substantive individuals. For when the class  $X$  is said to be the remainder to the class  $Y$ , part of what is meant is that no individual comprised in the one class is identical with any individual comprised in the other. Now this relation of non-identity has been repeatedly confounded with that of negation; so much so, that an important school of philosophy seems to hold that diversity or non-identity involves *prima facie* a contradiction; in other words, that the togetherness of non-identical substantives in the universe of reality involves the joint affirmation and negation of one and the same predicate.

The correspondences between adjectival relationships and class relationships will now be shown by first taking each of the four universals  $A, A', E, E'$ . Thus:

<sup>1</sup> The reversed confusion is committed when adjectival predications are spoken of as 'co-exclusive' or 'co-exhaustive' instead of 'co-disjunct' or 'co-alternate.'

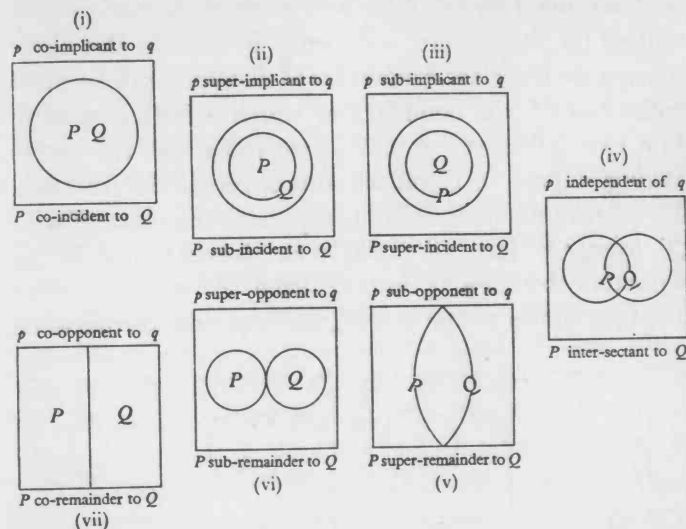
|                     | Adjectival Relations         | Distributive Relations          | Class Relations              |
|---------------------|------------------------------|---------------------------------|------------------------------|
| Direct Implicative  | $p$ implies $q$              | $A$ . Every $p$ is $q$          | $A$ included-in $Q$          |
| Counter Implicative | $p$ is implied by $q$        | $A'$ . Every $q$ is $p$         | $A$ includes $Q$             |
| Disjunctive         | $p$ and $q$ are co-disjunct  | $E$ . No $p$ is $q$             | $P$ and $Q$ are co-exclusive |
| Alternative         | $p$ and $q$ are co-alternate | $E'$ . Everything is $p$ or $q$ | $P$ and $Q$ are co-haustive  |

Since  $O$ ,  $O'$ ,  $I$ ,  $I'$  respectively contradict  $A$ ,  $A'$ ,  $E$ ,  $E'$ , we derive the following combinatory results:

|                                    |                                                                                             |                               |
|------------------------------------|---------------------------------------------------------------------------------------------|-------------------------------|
| (i) $p$ is co-implicant to $q$     | ( $A$ and $A'$ .) Every $p$ is $q$ and Every $q$ is $p$                                     | $P$ is co-incident to $Q$     |
| (ii) $p$ is super-implicant to $q$ | ( $A$ and $O'$ .) Every $p$ is $q$ and Not every $q$ is $p$                                 | $P$ is sub-incident to $Q$    |
| (iii) $p$ is sub-implicant to $q$  | ( $A'$ and $O$ .) Every $q$ is $p$ and Not every $p$ is $q$                                 | $P$ is super-incident to $Q$  |
| (iv) $p$ is independent of $q$     | ( $O$ and $O'$ .) Some but not every $p$ is $q$ and Some but not every non- $p$ is non- $q$ | $P$ is inter-secant to $Q$    |
| (v) $p$ is sub-opponent to $q$     | ( $E'$ and $I$ .) Everything is either $p$ or $q$ and Something is both $p$ and $q$         | $P$ is super-remainder to $Q$ |
| (vi) $p$ is super-opponent to $q$  | ( $E$ and $I'$ .) Nothing is both $p$ and $q$ and Not everything is $p$ or $q$              | $P$ is sub-remainder to $Q$   |
| (vii) $p$ is co-opponent to $q$    | ( $E$ and $E'$ .) Nothing is both $p$ and $q$ and Everything is either $p$ or $q$           | $P$ is co-remainder to $Q$    |

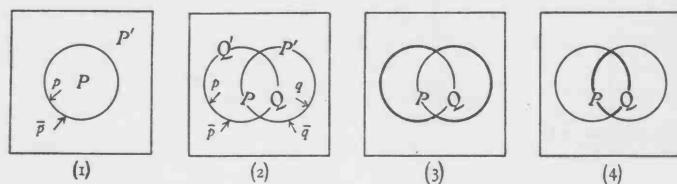
Here the class-relationships must be compared and contrasted with the adjective-relationships. In particular a 'super'-relation for the adjectives always yields a 'sub'-relation for the classes—illustrating the general principle that a *more determinate* connotation yields a narrower denotation, and a *less determinate* connotation

yields a wider denotation. The above seven relations may be at once expressed on Euler's scheme. Thus:



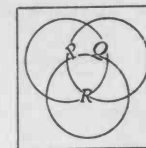
These diagrams, due to Euler, illustrate the adaptation of diagrammatic representation to propositions expressed, as above, in terms of classes. The employment of diagrams in Logic requires some special explanation. A class is represented by a closed figure, while anything that is comprised in the class may be represented by a point within this figure; and anything not comprised in the class, by a point outside the figure. It is further requisite that the all-inclusive class (otherwise called the universe) whether restricted or unrestricted should be represented also by a closed figure within which all the specific classes adjectively delimited should fall. Thus the boundary line may be taken to represent the adjective by which the scope of the class is determined,

while the area within this boundary-line represents the class itself. In the figure below the class  $P$  is represented as determined by the adjective  $p$ ; the universe is represented by the square, and what is outside the circle represents the class-remainder to  $P$ , which will be symbolized as  $P'$ , the boundary of which is indicated by  $\bar{p}$ . For two adjectives  $p$  and  $q$ , we must use two areas having a part in common with a remainder to both. The thickened outline in (3) separates the class ' $P$  with  $Q$ ' from ' $P$  into  $Q$ '; and the thickened line in (4) separates the class ' $P$  into  $Q$ ' from ' $P$  with  $Q$ .' The diagram shows to the eye the correspondences between

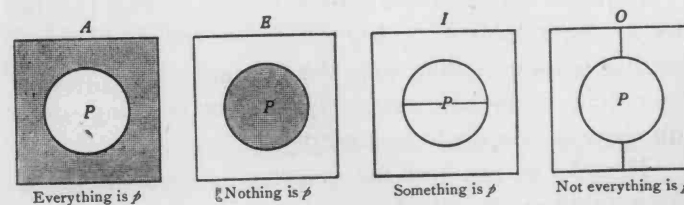


the adjectival and the class functions; viz., that the class determined by the adjectival *alternation* ' $p$  or  $q$ ' is the class ' $P$  with  $Q$ ,' and that determined by the adjectival *conjunction* ' $p$  and  $q$ ' is the class ' $P$  into  $Q$ .' These diagrams, first employed by Dr Venn, do not represent any proposition, but the framework into which propositions may be fitted. Thus it is shown, for instance, that, using two determining adjectives— $p$  and  $q$ —the universe is divided into  $2 \times 2$  classes, namely:  $P$  into  $Q$ ,  $P$  into  $Q'$ ,  $P'$  into  $Q$ , and  $P'$  into  $Q'$ , determined respectively by the adjectival conjunctions ' $p$  and  $q$ ,' ' $p$  and  $\bar{q}$ ,' ' $\bar{p}$  and  $q$ ,' ' $\bar{p}$  and  $\bar{q}$ .' Again: taking three determining adjectives  $p$ ,  $q$ ,  $r$ , we must draw three closed figures

in such a way that every resulting sub-class shall be represented; namely the  $2 \times 2 \times 2$  classes  $P$  into  $Q$  into  $R$ ,  $P$  into  $Q$  into  $R'$ ,  $P'$  into  $Q$  into  $R$ ,  $P'$  into  $Q$  into  $R'$ ,  $P$  into  $Q'$  into  $R$ ,  $P$  into  $Q'$  into  $R'$ ,  $P'$  into  $Q'$  into  $R$ ,  $P'$  into  $Q'$  into  $R'$ , as determined by the corresponding adjectival conjunctions.



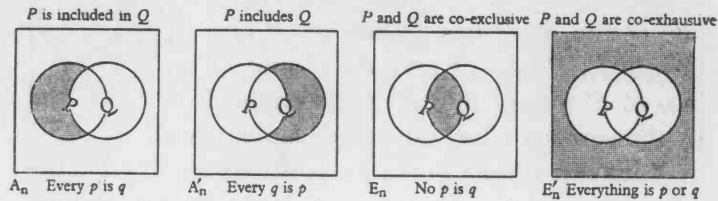
Into these frameworks propositions are fitted in the following manner. Beginning with a single determining adjective  $p$ , consider the four propositions:  $A$ . Everything is  $p$ ;  $E$ . Nothing is  $p$ ;  $I$ . Something is  $p$ ;  $O$ . Not everything is  $p$ . These four propositions can be expressed in terms of the classes  $P$  and  $P'$  thus:  $A$ .  $P$  exhausts the universe or  $P'$  is empty;  $E$ .  $P$  is empty or  $P'$  exhausts the universe;  $I$ .  $P$  is occupied or  $P'$  does not exhaust the universe;  $O$ .  $P$  does not exhaust the universe or  $P'$  is occupied. The import of each of these propositions may therefore be expressed by means of the opposite conceptions of *occupied* and *empty*: crowded horizontal shading will be used to indicate empty, and a single straight line to indicate occupied:



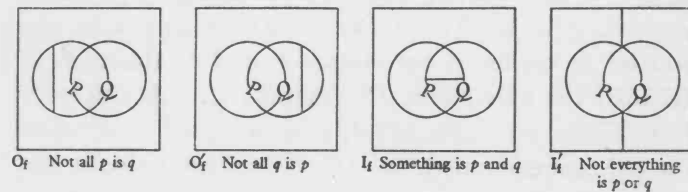
This shows that the universals may be expressed as denying and the particulars as affirming occupation. Taking now two adjectives  $p$  and  $q$ , we have eight distinct propositions  $A$ ,  $A'$ ,  $E$ ,  $E'$ ,  $O$ ,  $O'$ ,  $I$ ,  $I'$ , where



again the universals deny and the particulars affirm the occupation of certain sub-classes.



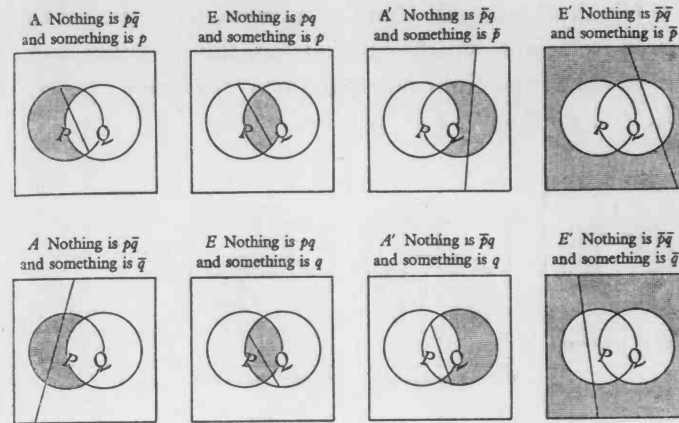
These are respectively contradicted by



In this system the strict relation of contradiction is indicated as of  $O$  to  $A$ ,  $O'$  to  $A'$ ,  $I$  to  $E$ ,  $I'$  to  $E'$ , by the single straight line and the shading occurring in the same sub-class, namely  $PQ'$ ,  $P'Q$ ,  $PQ$ ,  $P'Q'$ .

It must be observed that in Venn's system (1) the circles are drawn in every case as overlapping one another without exhausting the universe, and (2) that the specific proposition is represented by marking some sub-class as occupied or as empty.

Next let us combine the diagrams representing the several universals, which contain *no* instantial affirmation, with the diagram representing 'Something is  $p$ ' for  $A$  and  $E$ , and 'Something is not- $p$ ' for  $A'$  and  $E'$ ; or again with the diagram representing 'Something is  $q$ ' for  $A$  and  $E'$ , and 'Something is  $q$ ' for  $A'$  and  $E$ .



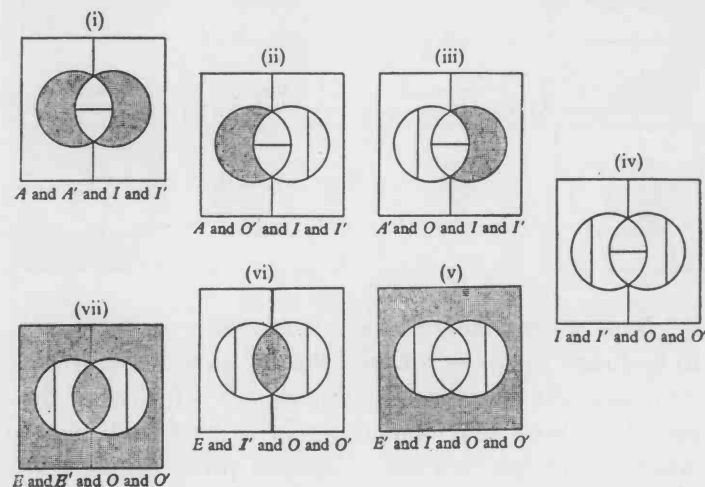
Now since a single line means that something is to be found in one *or* other of the two sub-classes which it crosses, and since the shading denies it for *one* of the two, it follows that something is to be found in the other. Thus the first four diagrams prove to the eye respectively that:

From 'Every  $p$  is  $q$  and Something is  $p$ ' we can infer  $I$ . 'Some  $p$  is  $q$ '  
 From 'No  $p$  is  $q$  and Something is  $p$ ' we can infer  $O$ . 'Some  $p$  is non- $q$ '  
 From 'Every  $q$  is  $p$  and Something is non- $p$ ' we can infer  $I'$ . 'Some non- $p$  is non- $q$ '  
 From 'Every-thing is  $p$  or  $q$ ' and 'Something is non- $p$ ' we can infer  $O'$ . 'Some non- $p$  is  $q$ '

And similarly for the second four, where the affirmative instantial involves non- $q$  or  $q$ .

These inferences illustrate the general principle that in order to infer a proposition giving instantial affirmation, we must have a premiss giving instantial affirmation. From  $A_n$  alone we cannot infer  $I_f$ , but from  $A_n$  jointly with 'Something is  $p$ ' we can infer  $I_f$ .

Finally let us use Venn's diagrams to represent the seven relations which result from the possible combinations of the eight elementary propositions  $A_n, A_n', E_n, E_n', I_n, I_n', O_n, O_n'$ .



In comparing this scheme with that of Euler (previously given) two points arise. Euler draws these figures from the outset so as to represent the class-relationships both as regards instantial affirmations and instantial denials, so that the figures directly express propositional information. But in Venn both these factors in the proposition have to be specifically marked and in order to represent a completely determined class-relationship all the four sub-classes must be marked. In spite of this apparent difference, an optical comparison of this last scheme with the Eulerian scheme on p. 149 will disclose their essential equivalence. The practical distinction, however, remains that in Euler's scheme each uncom-

pounded categorical must be represented by an *alternative* of figures, viz.:

$A$  by (i) or (ii); and  $O$  by (iii) or (iv) or (v) or (vi) or (vii)  
 $A'$  by (i) or (iii); and  $O'$  by (ii) or (iv) or (v) or (vi) or (vii)  
 $E$  by (vi) or (vii); and  $I$  by (i) or (ii) or (iii) or (iv) or (v)  
 $E'$  by (v) or (vii); and  $I'$  by (i) or (ii) or (iii) or (iv) or (vi)  
 and conversely,

each diagram represents a *conjunction* of propositions

(i) =  $A$  and  $A'$ , (ii) =  $A$  and  $O'$ , (iii) =  $A'$  and  $O$ ,  
 (iv) =  $I$  and  $I'$  and  $O$  and  $O'$ ,  
 (v) =  $E'$  and  $I$ , (vi) =  $E$  and  $I'$ , (vii) =  $E$  and  $E'$ .

On the other hand Venn's diagrams represent each of the uncompounded propositions by its appropriate 'marking' of the proper sub-class, and are thus immediately adapted to the conjunction of two or more affirmatively or negatively instantial pieces of information.

§ 8. All the above inferential operations are performed upon adjectival factors, these occurring always as predicates in a principal or subordinate clause; and, as is impressively brought out in the so-called 'existential' formulation of the proposition, a residual substantival factor always remains in the subject, though for linguistic convenience it may appear also in the predicate. The importance of this feature may have been obscured owing to the complicated detail with which the inferences have been treated; and, in conclusion, it is therefore to the point to emphasize the connection between the account of inference in this chapter and that of the functioning of substantive and adjective given in Chapter I.

## CHAPTER X

## EXISTENTIAL, SUBSISTENTIAL AND NARRATIVE PROPOSITIONS

§ 1. BEFORE directly approaching the topic to be dealt with in this chapter, it will be necessary to consider the general question of the classification of propositions. In previous chapters, several classifications of propositions under different *fundamenta divisionis* have been given: for example they have been divided into simple and compound, the latter being subdivided into conjunctive and composite; and again into uncertified and certified, the latter being subdivided into formally and experientially certified; even the distinction between true and false yields an exclusive and exhaustive division of propositions. On the other hand, many well-known so-called classifications of propositions break the purely formal rules of logical division, in that the sub-classes are not mutually exclusive, and often can hardly be regarded as collectively exhaustive. The most notorious example of this is the classification of propositions upon which Kant based his enumeration of the categories, and which comprised such sub-classes as singular, particular, universal, affirmative, negative, hypothetical, categorical, assertoric and problematic. Regarded as a classification of propositions this involves a flagrant violation of the formal rules of division; for a categorical proposition may be singular or universal, negative or affirmative, problematic or assertoric, etc. What is of real logical value, and was indeed intended

by Kant, is a classification not of propositions, but of the several formal relations which may enter within the structure of a proposition more or less simple or complex. For example, 'If the American harvest is bad, the European prices of corn are high' is properly enough denominated hypothetical because the central or principal relation asserted is that of implication; but further analysis discloses the categorical nature of the two implicationally related clauses, and the universality of the statement as understood to refer to any or every year; and furthermore such relations as contemporaneity and causality may be taken as implicitly asserted in such a proposition. Of distinctions amongst the different forms of relation that may enter into a proposition we may select as one of the most important that between the relation of characterisation and the relation of implication, which, properly speaking, should take the place of the distinction amongst propositions between categorical and hypothetical. Of these two relations—characterisation and implication—the former holds only of an adjective to a substantive, the latter only of one proposition to another proposition. Again, the distinctions of modality are not, properly speaking, distinctions between propositions, but distinctions between the different adjectives that can be significantly predicated of propositions. In short the sole logical purport of a so-called classification of propositions is, by means of an *analysis* of propositions of various forms of complexity, to disclose the different modes in which their components are bound into a unity.

With special reference to the topic of this chapter we may pass to such logicians as Lotze, Bosanquet

and others who have attempted to classify propositions on philosophical rather than purely formal principles. In particular Sigwart has distinguished propositions under three—not necessarily exclusive or exhaustive—heads corresponding to what, in the title, we have called existential, subsistential and narrative.

§ 2. We proceed then to examine in the first place what is meant by an existential proposition. The most general and appropriate sense in which the word 'existential' is predicated of a proposition is where the proposition refers to existence in that narrower sense in which existence is distinguished from subsistence, as two sub-divisions of reality. Thus the proposition '3 plus 4 equals 7' must be regarded not as existential, but as subsistential, if that terminology be permitted. From an examination of the illustrations given by philosophers of the existent (as distinct from the subsistent) it may be gathered that the term is equivalent to that which is manifested in time or space. This interpretation may be justified by considering the etymology of the word 'existent' which is closely connected with 'external,' and is further confirmed by the fact that the typical so-called external relations are temporal or spatial. On the other hand it has been maintained, for example, that the number 3, or the relation of equality between 3 plus 4 and 7, or the relation of causality subsists rather than exists. If this conception is to be generalised what subsists is primarily an adjective, whether ordinary or relational; whereas what in the more exact sense may be said to exist is a substantive proper. We may therefore regard the terms 'existent' or 'substantive proper' as meaning 'what is manifested in time or

space.' Thus an existential proposition is distinguished from a subsistential proposition in that the latter makes predications about adjectives (including propositions) as such. It may, however, be maintained that such a proposition as '3 plus 4 equals 7' should properly be interpreted as existential on the ground that it applies to all possible existent groupings of classes numbered 3, 4, and 7; or again that the proposition 'Unpunctuality is irritating' is existential on the ground that it means nothing more nor less than that 'All unpunctual arrivals are irritating,' where the term arrival, with its implicit temporal and spatial reference, obviously stands for an existent. Or yet again that the proposition 'Heat causes wax to melt' is existential on the ground that it merely expresses the universal proposition that 'All cases in which heat enters wax, are cases in which the wax is melted,' where merely temporal and spatial relations of possible occurrences are involved. If then a subsistential proposition is to be distinguished from an existential, it must be on the ground that propositions in which the *explicit* predications concern adjectives or relations, have a special significance beyond what they undoubtedly imply existentially<sup>1</sup>.

§ 3. But turning from the more philosophical to the strictly formal usage of the term existential, we find that by such logicians as Venn, Keynes and Russell, existential and subsistential propositions are indifferently denominated *existential*, and that the term existential is used without any reference to the *substance* of the proposition, but rather to a certain mode in which

<sup>1</sup> This question will be treated in more detail in a subsequent chapter.

any *general* proposition (particular or universal) may be *formulated*. This entirely distinct and peculiar use of the term 'existential' has given rise to endless confusion; and, on this account, the term should be entirely discarded and replaced by some such term as *instantial*, or, more accurately, *indeterminately instantial*. At this point we must explain the distinction between determinately instantial and indeterminately instantial. While the former corresponds roughly to narrative propositions, of which we shall treat later, the latter are most naturally prefixed by the phrase 'there is' or 'there are'; e.g. there is a God; there are horses; there are no sea-serpents; there is an integer between 3 and 5; there are prime integers between 4 and 15; there is no integer between 3 and 4. Of these, the first three would be called existential, in the philosophical sense, the last three subsistential. A minor distinction amongst such indeterminately instantial propositions, the disregard of which has not infrequently led to confusion, is that between the *affirmatively instantial* and the *negatively instantial*. In short, the essential nature of a particular or of a universal proposition is expressed by formulating the former as affirmatively instantial, and the latter as negatively instantial<sup>1</sup>.

The further development of the topic of such propositions, or rather of such propositional formulations, requires us to introduce the phrase '*universe of discourse*,' to which frequent reference is made in formal expositions of the so-called existential import of propositions. There are two applications of this phrase, which demand different criticisms. One quite harmless

<sup>1</sup> See Chapter VIII.

application of the expression 'universe of discourse,' points merely to the familiarly *elliptical* nature of conversation. Thus the reference of such a proposition as 'All voters are males' is understood to be limited, say, to the present time (1914), the English nation, and election to Parliament. In spite of the fact that some nations now and all nations will confer the franchise upon women, in spite of the fact that for the Board of Guardians and other offices women take part in voting, the proposition 'all voters are males' is perfectly intelligible in its context. The phrase, though elliptical—like all phrases in discussion or conversation—does not require the explicit introduction of every well-understood qualification. In our view, therefore, logicians have unnecessarily paraded this application of the notion of a universe of discourse, where it means merely that the ordinary reader is expected to supply the restrictions intended by the writer. A limited universe understood as indicating the subject-matter of a single work, such as geometry, which refers exclusively to spatial figures, illustrates the same simple relation to the universe as a whole. Understood in this general sense, the universe of discourse has to *the* universe the relation of part to whole, and the notion is certainly harmless if trivial.

But the other application of the phrase requires more serious criticism. Here 'the universe of discourse' is presented to the reader, not as *inside*, but as *outside* what is commonly called *the* universe. In this usage the phrase seems to imply that there are several universes related to one another as Europe is to Africa rather than as France is to Europe, taking Europe in both cases to stand for *the* universe. The distinction



between the two uses of the phrase is evident when we pass from such an example as 'All voters are males' to 'Some fairies are malevolent' which well illustrates the use now under consideration. The former is understood to refer to a limited part of the universe of persons, whereas the latter refers to no part whatever of this universe, and on this account is said to be concerned with the 'universe of imagination' conceived as outside and separate from the universe of reality. Now if, when speaking of specific universes, such as the universe of imagination, the universe of ideation, and the universe of physical reality, we meant merely universes comprising images, ideas, or physical realities, then all these three are in the strictest sense included within the one single universe of existents, to which they are related merely as parts to whole. Anything that is comprised in the universe of images must be an image; anything that is comprised in the universe of ideas must be an idea (both of these being *psychical*); and similarly, anything comprised in the universe of physical realities must be physical. What logicians seem to have confused, and requires only common sense to distinguish, is between a horse and either the idea of a horse or the image of a horse; and accordingly a proposition about horses is concerned with different material from any proposition about ideas or images of horses. When then a proposition is spoken of as being false in the universe of reality and yet true in the universe of imagination or ideation, this involves the tacit assertion that the *same* proposition can be both true and false; whereas in fact the contents of the two propositions, one of which is said to be true and the other false, are different. The affirmation

we images  
psychical

or denial that there are sea-serpents is different from the affirmation or denial that there are images of sea-serpents; which again is different from the affirmation or denial that there are ideas of sea-serpents. It is absurd to say that the same things exist in one universe and do not exist in another: wherever this appears to be the case the things asserted or denied to exist are different. What is here said of sea-serpents holds equally of horses or of dragons; as regards the latter, it is supposed that because dragons are acknowledged not to exist in the universe of physical reality, there must be some universe in which they do exist in order that we may intelligently use the term 'dragon.' Now it is a purely psychological question whether at this or that moment of time an image of a horse or equally of a dragon is in course of mental construction; it may be that we may intelligently read or think about dragons or horses without mentally constructing any images of such creatures. Properly speaking there is no such thing as *the* image of a horse or *the* image of a dragon, because the constructing of images by one person at one time is, as an occurrence, distinct from such a construction by another person at another time, however closely these images may agree with one another in character. Hence, if existence is predicated of any image of this or that kind, it must be remembered that by existence is here meant manifestation in time, and that therefore there exist as many images of any kind of thing as there are occurrences of the constructive act.

What holds of images holds, strictly speaking, also of ideas, though not so obviously; the existence or non-existence of the idea of any object, if idea stands



for mental process, must mean the occurrence or non-occurrence of an act of thinking about the object during this or that period of time. The term 'idea,' however, may be understood in a less literally psychological sense: thus intelligently to entertain a proposition in thought would seem to entail our entertaining ideas corresponding to the several terms in the proposition. But in this connection we may refer to Mill's pronouncement in regard to the import of propositions in relation to ideas. His dictum is that propositions are not about ideas, but about things; and by this he intended to assert that a proposition is concerned with the things which it expressly talks of, and not with any mental process that may be involved in the assent to or understanding of the proposition. In short, although any genuine act of assertion requires as a preliminary process the understanding of the terms and combination of terms that constitute a proposition, yet it is not this process to which the proposition refers. This, of course, holds, whether the matter of the proposition is physical reality or mental reality: we must understand what is meant by the association of ideas or by an emotion of anger in a psychological proposition, just as we must understand what is meant by dragons or horses in propositions describing such creatures; while, on the other hand, the propositions in neither case are concerned with these processes of understanding.

§ 4. We next proceed to consider in what sense truth and falsity can be predicated of propositions such as 'Some fairies are malevolent' or 'No Greek gods are without human frailties.' These may be otherwise rendered: 'There are malevolent fairies,' 'There are no

Greek gods without human frailties.' If these be taken literally, as merely primary propositions, nothing can be said but that the first is necessarily false and the second necessarily true, because there are actually no fairies and no Greek gods in the real universe. But within the real universe there are to be found *descriptions* of fairies and of Greek gods in stories or legends, and hence it may be *true* that some fairies have been *described* as malevolent, and it may be *false* that no Greek gods have been *described* as without human frailties. If then we distinguish these secondary propositions—to be recognised as such by the introduction of the word 'describe'—from the original primary propositions, the establishment of their truth or falsity is seen to depend upon special evidence. The universe of descriptions is simply part of the universe of reality; indeed it seems strangely to have escaped logicians that books and the persons who wrote them belong to one real world, and that therefore the universe to which we refer for verification of propositions concerning the descriptions of fairies or of Greek gods is simply and precisely the same universe as that to which we refer for verification of propositions concerning Frenchmen and geologists. It is therefore evident that, only when we have transformed such primary propositions into their secondary correspondents, any question of interest arises as to their truth or falsity. This contention finally forbids us to speak of various universes of discourse which are outside the one universe of reality. The briefest mode of indicating the peculiarity of propositions of the type illustrated is to say quite simply that they are elliptical; not elliptical in the sense of limiting the subject-term to a narrower sphere

*This is the  
correct view*

included in the universe, but elliptical in the sense of being expressed as primary propositions and understood as secondary. Thus our first example should properly be expressed 'Story-books describe some fairies as being malevolent,' and our second 'Homer describes all the Greek gods as subject to human frailties'; and in these transformed shapes the propositions are seen at once to be verifiable in exactly the same way as any other propositions; namely by reference to the one real universe of books and persons.

§ 5. We pass now to the logical significance of the term narrative in its application to propositions. The notion of a narrative proposition is not restricted to the type of proposition characteristic of a work of fiction or history, since it includes statements made in ordinary conversation etc., where there may be no intention to develop the account of an incident into a connected story. Moreover histories and novels are composed of others besides narrative propositions—the non-narrative propositions being generally what we may call comments on the incidents, characters, situations or emotions described. Novels (or even histories) might indeed be classified according as their narrative or commentary elements predominate; compare for example Scott with Thackeray, or S. R. Gardiner with Macaulay. A narrative proposition may be more precisely defined as one whose subject-term is prefixed by introductory or referential applicatives; whereas non-narrative propositions are prefixed by such distributives as 'every,' 'some' or similar phrases. Now distributives serve as predesignations of adjectivally significant subjects, while commentary propositions may be distinguished from such

narrative propositions as may happen to use subjects containing an adjectival element because in the *latter* case the adjective has no general reference. For example, in the course of a narrative the proposition may occur 'A shabbily-dressed gentleman entered the room,' and this may be followed later on by 'the shabby gentleman withdrew,' where the adjective 'shabby' enters in the context merely first as introductory and later as referential. In fact it is not by the consideration merely of the grammatical structure of a sentence, but rather by the logical nexus of the propositions that the distinction can be established; the narrative and non-narrative elements in any literary work being not necessarily expressed in separable sentences. Hence the reader may easily pick out the commentary elements, these being recognisable by their reference to persons and things in general as distinct from the persons and things entering directly into the plot.

My account of narrative propositions covers a wider range than is apparently intended by Sigwart; but for both of us the distinction between narrative and non-narrative rests upon that between the substantive and the adjective. In Sigwart's definition the subject in the narrative proposition is merely substantival, while the subject as well as the predicate in the non-narrative proposition contains adjectival elements. My application of the term 'narrative' on the other hand, includes cases in which the subject term may contain an adjectival element the significance of which is purely introductory or referential.

§ 6. A new problem, bearing upon the existential import of propositions is raised when we contrast

included in the universe, but elliptical in the sense of being expressed as primary propositions and understood as secondary. Thus our first example should properly be expressed 'Story-books describe some fairies as being malevolent,' and our second 'Homer describes all the Greek gods as subject to human frailties'; and in these transformed shapes the propositions are seen at once to be verifiable in exactly the same way as any other propositions; namely by reference to the one real universe of books and persons.

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fictitious with historical narratives. We may take for illustration the proposition 'Mr Pecksniff is a hypocrite' and first ask what is meant by Mr Pecksniff. Now a provisional answer to this would be 'A certain architect, living near Salisbury, in the beginning of the nineteenth century.' The term by which we have replaced Mr Pecksniff seems to have an obvious reference to the universe of reality, and more particularly to the universe of things happening and existing in time and place. But the question as to whether there was any architect in Salisbury at that time would be irrelevant, and therefore the proposition would appear not to be about any architect then living near Salisbury. The difficulty here points to a peculiarity in the use of the predesignation 'a certain.' If for Pecksniff we had substituted 'some architect' instead of 'a certain architect,' the proposition 'Some architect living then near Salisbury was a hypocrite' would have been amenable to the ordinary modes of verification. But the form of statement 'A certain architect was a hypocrite' appears not to represent a proposition, inasmuch as it cannot be either affirmed or denied, since the architect to whom the writer refers is not indicated. What holds then of the reader or hearer of such a proposition does not hold of the writer or speaker<sup>1</sup>. Though the hearer is unable to give the direct contradictory of the proposition, yet the speaker may propound the two alternatives

<sup>1</sup> According as logicians exclusively interpret propositions from the point of view of the speaker (writer) or hearer, they are to be classed respectively as conceptualists or nominalists. The difference between these two points of view lies at the root of many logical controversies.

that a certain *S* is or is not *P*, provided that he has his own individual means of identifying the *S* to whom he is referring in thought. In fact the most common usage of the phrase 'a certain' involves deliberate concealment for various harmless purposes on the part of the speaker. Thus, when I say 'A certain boy now in this room has stolen my purse,' I deliberately preclude any hearer from strictly contradicting or agreeing with me, though of course he could deny the proposition by asserting not the contradictory but a contrary, namely: 'No boy in this room has stolen the purse.'

As regards a narrative, fictitious or historical, however, where any substantival reference must always be interpreted in accordance with its nexus with the introductory 'a certain' (coupled or not with a proper name), the writer and reader are so far in the same position that neither the one nor the other is concerned with the question of ultimate identification. The referential 'the' is prefixed to an object identical with that to which the introductory 'a' was first prefixed, but outside and beyond this nexus there is no further possibility of identification. Hence the whole body of propositions in a fictitious narrative is not entertained with a view to the consideration of their truth or falsity, and might be called pure suppositions. The scholastic logicians introduced the phrase 'suppositio materialis' which would illustrate the sense in which 'supposition' has just been used; but modern logicians have interpreted this phrase as equivalent to what they call the universe of discourse.' It is obvious, however, that the two conceptions are totally distinct, inasmuch as the former consists of

classes of propositions included in the universe of all possible propositions, whereas the latter consists of classes of substantives included in the universe of all possible substantives. In contrasting a work of fiction with an historical work, the propositions laid down in the latter are put forward as to be accepted as true on the authority of the writer. But in both cases, whether history or fiction, it still holds that there is no means for ultimately identifying the characters introduced either on the part of the reader or the writer; we can only say that in history it is believed that these characters are identifiable with persons who have actually existed, whereas in fiction no such belief is involved.

§ 7. Within propositions which are fictitious, the distinctions between those which introduce beings to which there are, and to which there are not, similar beings in the world of reality, gives rise to a further problem<sup>1</sup>. Thus we may contrast the various statements about the architect Mr Pecksniff in *Martin Chuzzlewit* with the various statements that might be made about the fairy Puck in a fairy-tale. It will be noted that in the former case the general class (architect) to which reference is made, actually exists, whereas in the latter the class (fairy) to which reference is made, does not exist; while neither the individual Pecksniff nor the individual Puck does or ever did exist. This immediately gives rise to the question of the distinction or

<sup>1</sup> Of course there is a further distinction between fictions (novels, dramas) which describe characters and incidents such as *might* occur in the real world, and fairy-tales (myths, legends) which give descriptions such as never can occur, in that judgments as to their naturalness or 'realism' in the former case are more often relevant than in the latter.

relation in the significance of the word 'exist' as applied, firstly to a class, and secondly to an individual. We have above pointed out the peculiar and comparatively modern application of the word 'exist' on the part of formal logicians, who express the proposition: 'There are architects' in the form 'The class architect exists'; and the proposition 'There are no fairies' in the form 'The class fairy does not exist.' A more precise formulation of these propositions is obtained by taking *C* to stand illustratively for any class, the affirmation of whose existence is thus rendered: 'There is at least one individual, say *P*, which is comprised in *C*'; or rather, since a class is determined by connotation: 'There is at least one individual, say *P*, which is characterised by the conjunction of adjectives constituting the connotation of the name *C*.' Now here, I maintain, that the symbol *P* stands for a proper or uniquely descriptive name, and hence that the conception of the existence of a *class*—indicated by a connotative name—requires the conception of the existence of an *individual*—indicated by a proper or uniquely descriptive name. Now we may agree that there is no such individual as Pecksniff, and that there is no such individual as Puck; although in the first case the *class* 'architect'—which might be used in the description of Pecksniff—would be said to exist, while the class 'fairy'—which might be used in the description of Puck—would be said not to exist. If then we brought forward Sir Christopher Wren and Mr Pecksniff as instances of architects, or Oliver Cromwell and Mr Pecksniff as instances of hypocrites, would this substantiate the affirmation that there are at least *two* individuals comprised in the class



architect, or in the class hypocrite? If this question is answered in the negative it must be on the ground that, in some sense of the term 'exist' which is not appropriate to *classes*, Mr Pecksniff does not and never did exist, and hence he cannot count as one when we are enumerating the members comprised in any given class. Furthermore, since the numerical predication 'at least one' is highly indeterminate and could be in this or that case replaced by the relatively determinate 'at least  $n$ ' where  $n$  stands for this or that number, the affirmation that 'the class  $C$  exists' is only a special and less determinate case of the affirmation that 'the class  $C$  comprises at least  $n$  items,' and the number  $n$  cannot be counted as such unless all the  $n$  items *exist*. The conclusion therefore follows that the sense of the word 'exist' when predicated of a class is dependent upon that of the word 'exist' when predicated of an item or individual indicated by a proper or uniquely descriptive name<sup>1</sup>.

<sup>1</sup> This contention is directed against the position held in the *Principia Mathematica*, where  $E!$  is ultimately defined in terms of  $\mathfrak{A}$ , whereas in my view  $\mathfrak{A}$  is to be ultimately defined in terms of  $E!$

## CHAPTER XI

## THE DETERMINABLE

§ 1. IN this chapter we propose to discuss a certain characteristic of the adjective as such, which perhaps throws the strongest light upon the antithesis between it and the substantive. Here it will be apposite to consider the traditional account of the principles of logical division where a class (of substantives) is represented as consisting of sub-classes. This process is governed by the following rules: (1) the sub-classes must be mutually exclusive; (2) they must be collectively exhaustive of the class to be divided; (3) division of the class into its co-ordinate sub-classes must be based upon some one 'fundamentum divisionis.' The first two of these rules may be said to be purely formal, and do not raise any problem of immediate interest; but the technical term *fundamentum divisionis*—though perhaps readily understood by the learner—is actually introduced without explicit account of its connection with, or its bearing upon, ideas which have entered into the previous logical exposition. To illustrate the notion we are told, for instance, that, when a class of things is to be divided according to colour, or to size, or to some other aspect in which they can be compared, then the colour, size, or other aspect constitutes the *fundamentum divisionis*. Now although, grammatically speaking, words like colour and size are substantival, they are in

See Sturt's "Henrietta Herz" Lecture, in *Proc. Brit. Acad.* - Vol. 8 (1921-22). pp. 12 ff.



fact abstract names which stand for adjectives; so that the fundamentum divisionis is, in the first place, an adjective, and in the second, an adjective of the particular kind illustrated by 'colour' when considered in its relation to red, blue, green, etc. Superficially this relation appears to be the same as that of a single object to some class of which it is a member: thus two such propositions as 'Red is a colour' and 'Plato is a man' appear to be identical in form; in both, the subject appears as definite and singular, and, in both, the notion of a class to which these singular subjects are referred appears to be involved. Our immediate purpose is to admit the analogy, but to emphasise the differences between these two kinds of propositions, in which common logic would have said we refer a certain object to a class.

I propose to call such terms as colour and shape *determinables* in relation to such terms as red and circular which will be called *determinates*; and, in introducing this new terminology, to examine the distinction between the relation of *red* to *colour* and the relation of *Plato* to *man*. To predicate *colour* or *shape* of an object obviously characterises it less determinately than to predicate of it *red* or *circular*; hence the former adjectives may be said negatively to be indeterminate compared with the latter. But, to supplement this negative account of the determinable, we may point out that any one determinable such as colour is distinctly other than such a determinable as shape or tone; i.e. colour is not adequately described as indeterminate, since it is, metaphorically speaking, that from which the specific determinates, red, yellow, green, etc., emanate; while

from shape emanate another completely different series of determinates such as triangular, square, octagonal, etc. Thus our idea of this or that determinable has a distinctly positive content which would be quite inadequately represented by the word 'indeterminate.' Further, what have been assumed to be determinables—e.g. colour, pitch, etc.—are ultimately *different*, in the important sense that they cannot be subsumed under some one higher determinable, with the result that they are incomparable with one another; while it is the essential nature of determinates under any one determinable to be comparable with one another. The familiar phrase 'incomparable' is thus synonymous with 'belonging to different determinables,' and 'comparable' with 'belonging to the same determinable'; not that this is the actual meaning of the terms, but that enquiry into the reason for the comparability or incomparability of two qualities will elicit the fact that they belong to the same or to different determinables respectively. This phrase 'belonging to' is also more usually used of a member of a class in relation to its class: we have, then, to contrast the significance of the relation 'belonging to' when applied in one case to a determinate and its determinable, and in the other to an individual and its class. If it is asked why a number of different individuals are said to belong to the same class, the answer is that all these different individuals are characterised by some the same adjective or combination of adjectives. But can the same reason be given for grouping red, yellow and green (say) in one class under the name colour? What is most prominently notable about red, green and yellow is that they are *different*, and even, as we may say, opponent

to one another; is there any (secondary) adjective which analysis would reveal as characterising all these different (primary) adjectives? In my view there is no such (secondary) adjective; in fact, the several colours are put into the same group and given the same name colour, *not* on the ground of any partial agreement, but on the ground of the special kind of difference which distinguishes one colour from another; whereas no such difference exists between a colour and a shape. Thus red and circular are adjectives between which there is no relation except that of non-identity or otherness; whereas red and blue, besides being related as non-identical, have a relation which can be properly called a relation of difference, where difference means more than mere otherness. What is here true of colour is true of shape, pitch, feeling-tone, pressure, and so on: the ground for grouping determinates under one and the same determinable is not any partial agreement between them that could be revealed by analysis, but the unique and peculiar kind of difference that subsists between the several determinates under the same determinable, and which does not subsist between any one of them and an adjective under some other determinable. If this is granted, the relations asserted in the two propositions 'Red is a colour' and 'Plato is a man,' though *formally* equivalent, must yet be contrasted on the ground that the latter but not the former is based upon an adjectival predication. For the latter is equivalent to predicating the adjective 'human' of 'Plato,' while, without denying that some adjectives may properly be predicated of (the adjective) red, yet the proposition 'Red is a colour' is *not* equivalent to predicating any adjective of red.

§ 2. Bearing in mind this distinction, the question arises whether what are called abstract names can be divided in the same way as concrete names into singular and general; in other words, whether adjectives can be divided into these two classes. The answer seems to be that adjectives can be divided into two classes more or less analogous to the singular and general which distinguish substantives, but that the two different kinds of adjectives are preferably distinguished as determinate and indeterminate. When, in considering different degrees of determinateness, the predication of one adjective is found to imply another, but not conversely, then the former we shall call a super-determinate of the latter and the latter a sub-determinate of the former. Thus the relation of super-determinate to sub-determinate means not only that the former is more determinate than the latter, but also that the predication of the former would imply that of the latter. A simple example can be taken from the determinable 'number': thus 7 is super-determinate to 'greater than 3'; the adjective 'greater than 3,' though not itself a *summum* determinable, may be called determinable, inasmuch as it is not merely indeterminate but capable of being further determined in the sense that it generates a definite series of determinates. To illustrate more precisely what is meant by 'generates'; let us take the determinable 'less than 4'; then 'less than 4' generates '3' and '2' and '1' in the sense that the understanding of the meaning of the former carries with it the notion of the latter. Now no substantive class-name generates its members in this way; take, for instance, 'the apostles of Jesus,' the understanding of this class-name carries with it the notion

i. 'men summoned by Jesus to follow him,' but it does not generate 'Peter and John and James and Matthew etc.,' and this fact constitutes one important difference between the relation of sub-determinate to super-determinate adjectives and that of general to singular substantives.

ii. § 3. Another equally significant difference is brought out by considering that aspect of substantive-classes in which—to use the terminology of formal logic—increase of intension is accompanied by decrease of extension. The phrase 'increase of intension' conjures up the notion of adding on one attribute after another, by the logical process called conjunction; so that, taking  $p, q, r$ , to be three adjectives, increase in intension would be illustrated by regarding  $p, q, r$  conjoined as giving a greater intension than  $p, q$ ; and  $p, q$  as giving greater intension than  $p$ . We have now to point out that the increased determination of adjectival predication which leads to a narrowing of extension may consist—not in a process of conjunction of separate adjectives—but in the process of passing from a comparatively indeterminate adjective to a comparatively more determinate adjective under the same determinable. Thus there is a genuine difference between that process of increased determination which conjunctively introduces foreign adjectives, and that other process by which without increasing, so to speak, the number of adjectives, we define them more determinately.

In fact, the foreign adjective which appears to be added on in the conjunctive process, is really not introduced from outside, but is itself a determinate under another determinable, present from the start, though

suppressed in the explicit connotation of the genus. We propose to use a capital letter to stand for a determinable, and the corresponding small letter with various dashes to stand for its determinates. Thus, in passing from the genus  $p$  to the species  $pq$ , we are really passing from  $pQ$  to  $pq$ ; or again the apparent increase of intension from  $p$  to  $pq$  to  $pqr$  is more correctly symbolised as a passing from  $pQR$  to  $pqR$  to  $pqr$ . In the successive process of dividing a summum genus into the next subordinate sub-genera, and this again into sub-sub-genera, the summum genus ought to be represented by a conjunction of determinables, say  $PQRST$ ; the genera next subordinate to this, by  $pQRST, p'QRST, p''QRST$ , etc., and the genera next subordinate to the first of these by  $pqRST, pq'RST, pq''RST$ , and so on down to the *infima species* represented by determinates. Thus:

|         |          |           |       |
|---------|----------|-----------|-------|
| $PQRST$ |          |           |       |
| $pQRST$ | $p'QRST$ | $p''QRST$ | ..... |
| $pqRST$ | $pq'RST$ | $pq''RST$ | ..... |

In this way we represent from the outset the nature of the ultimate individuals under the summum genus, as being characterisable jointly by the determinables  $PQRST$ , while any genus or species is represented by these same determinables, one or more of which are replaced by determinates. This meets a criticism which has often been directed against the formal account of the inverse variation of extension and intension, since we see now that the same *number* of adjectives should be used in giving the connotation of the wider as of the narrower class. To illustrate these symbols from a botanical classification of plants: let the determinable

$P$  stand for the number of cotyledons,  $Q$  for the disposition of the stamens,  $R$  for the form of the corolla,  $S$  for the attachment of the petals and sepals, and  $T$  for the divisibility of the calyx. Then  $PQRST$  represents the summum genus '*plants*' as describable under these five heads, but otherwise undetermined in character. Then  $p, p', p''$ , might stand respectively for having no cotyledons, having 1, and having 2, thus representing the defining characteristic of each of the three classes—acotyledon, monocotyledon, and dicotyledon—by the symbols  $pQRST, p'QRST, p''QRST$ . Again  $q, q', q''$ , might stand respectively for the stamens being under, around or upon the carpels, thus representing the three sub-divisions—hypogynous, perigynous, epigynous—of dicotyledons, as  $p''qRST, p''q'RST, p''q''RST$ . Taking regular and irregular to be the two possible forms of corolla, then the next sub-division under  $p''q'RST$  will be  $p''q'rST$  and  $p''q'r'ST$ . Again  $s$  and  $s'$  may stand respectively for separability and inseparability of the calyx and corolla, and yield the further sub-divisions, say  $p''q'rST, p''q'r'sT$ . The calyx may contain only one part or 3 or 4 or 5 or 6, and if these are represented respectively by  $t, t', t'', t''', t''''$ , a relatively determinate characterisation is finally symbolised by  $p''q'r's't''''$  say.

There are cases for which a modification of this general scheme is required. The cases are those in which one particular sub-division is definable by the *absence* of an element upon which the predication of other determinables depend, while in the sub-divisions co-ordinate with this the element in question is present. For example, the class of plants called acotyledons might be defined by the absence of any corolla, etc., and

hence such variations as that of the form of the corolla or the disposition of the stamens, etc., are inapplicable to this particular sub-division.

§ 4. Now adjectives under the same determinable are related to one another in various ways. One relational characteristic holds in all cases; namely that, if any determinate adjective characterises a given substantive, then it is impossible that any other determinate under the same determinable should characterise the same substantive: e.g. the proposition that 'this surface is red' is incompatible with the proposition 'this (same) surface is blue.' It has been usual to modify the above statement by adding the qualification—at the same time and at the same place; this qualification applies where the substantive extends through some period of time and over some region of space, in which case the existent substantive, having temporal or spatial parts, may be said to be extended. For this reason the qualification would perhaps better be attached to the substantive itself, and we should say that, where opponent adjectives are predicated, reference is made to different substantives, since any one part of an extended substantive is existentially *other* than any other part.

A second characteristic of many determinates under the same determinable is that the differences between different pairs of determinates can be compared with one another; so that if  $a, b, c$ , are three determinates, there are cases in which we may say that the difference between  $a$  and  $c$  is greater than that between  $a$  and  $b$ ; e.g. the difference between red and yellow is greater than that between red and orange. In this case the several determinates are to be conceived as

necessarily assuming a certain serial order, which develops from the idea of what may be called 'adjectival betweenness.' The term 'between' is used here in a familiar metaphorical sense derived from spatial relations, and is figuratively imaged most naturally in spatial form. Thus if *b* is qualitatively between *a* and *c*, and *c* qualitatively between *b* and *d*, and so on, the whole series has its order directly determined by the nature of the adjectives themselves. The further distinctions amongst series as interminable or as cyclic, and again of series of more than one order of dimensions, lead to logical complexities which need not be entered into here. Suffice it to say that this characteristic, which holds of so many determinates, gives significance to another well-known rule for logical division: *divisio non faciat saltum*: one meaning of which appears to be that we contemplate not merely enumerating a set of coordinate sub-classes, but enumerating them in a certain order. The rule prescribes that the order in which the sub-classes are enumerated should correspond to the order of 'betweenness' predicable of their differentiating characteristics.

The order of betweenness which characterises the determinates just considered may be either discrete or continuous. In the case of discrete series there is one determinate that can be assigned as next after any given determinate; but, in the case of a continuous series, a determinate can always be conceived as between any two given determinates, so that there are no two determinates which can be said to be next to one another in the serial order. It follows from this account of continuity that, between any two determinates which may be said to have a finite adjectival difference, may be interpolated

an indefinite number of determinates having a finite difference, and this number becomes infinite as the differences become infinitesimal. Amongst continuous series further differences between the interminable and the cyclic, and again between those of one or more order of dimensions, hold as in discrete series.

The reference here to determinables of higher or lower dimension requires explanation. Our familiar example of colour will explain the point: a colour may vary according to its hue, brightness and saturation; so that the precise determination of a colour requires us to define three variables which are more or less independent of one another in their capacity of co-variation; but in one important sense they are not independent of one another, since they could not be manifested in separation. The determinable colour is therefore *single*, though complex, in the sense that the several constituent characters upon whose variations its variability depends are inseparable.

§ 5. Returning to the conception of the absolutely determinate adjective, we have to note an important distinction between absolutely determinate and comparatively indeterminate predications. The distinction may thus be formulated: If, of two substantives the same *determinate* adjective can be predicated, then *all* the adjectives and relations definable in terms of the determinable, that can be predicated of the one, could be predicated of the other. But if, of two substantives the same *indeterminate* adjective can be predicated, then only *certain* of the adjectives and relations definable in terms of the determinable, that can be predicated of the one, can be predicated of the other. To illustrate



first the case of an indeterminate predication; let us take the numerical adjective 'greater than 7'; then of any collection of which this numerical adjective could be predicated, other adjectives such as 'greater than 5' and 'greater than 3' could also be predicated; but some collections that are 'greater than 7' such as the apostles, are greater than 11 and divisible by 4 for instance, whereas other collections that are 'greater than 7,' such as the muses, are less than 11 and are not divisible by 4: hence it is only *some* of the numerical adjectives that are predicable of the muses that are also predicable of the apostles, although the adjective 'greater than 7' is predicable of them both. Turning now to the case of *determinate* predication; if, instead of defining a collection by the indeterminate adjective 'greater than 7,' we had defined it by the determinate adjective 'twelve,' then any numerical adjective that is predicable of one collection of twelve, say the apostles, would be predicable of any other collection of twelve, say the months of the year or the sons of Israel; for example, 'greater than 11,' 'divisible by 4,' 'a factor of 96.' What we have here seen to hold of determinate and indeterminate number holds of any other determinable. The case of colour lends itself easily for illustration on account of the specific names which have been assigned to its determinates: thus, if the colours of two different objects are the same shade of yellow, then though these two objects may differ in any number of other respects such as shape and size, yet we may say that any colour-property of the one object will agree with the colour-property of the other; if the colour of one is more brilliant or less saturated than the colour of an orange,

then the same will hold for the colour of the other. In fact, whatever sensational determinable we take, whether it be colour, or sound, or smell, the determinate characterisations under any such determinable would lead to the same forms of generalisation that have been developed by science only in the sphere of quantity. It is agreed that in the sphere of sense perception, differences of quality strictly speaking hold only of the mental or sensational, and that the physical can only be defined in quantitative terms. Thus in the Weber-Fechner experiments the experient judges of equivalence or difference in the intensity or quality of his sensations, with which are correlated quantitative differences in the stimuli. The attempts that psychologists have made to discover formulae of correlation between the stimuli on the one hand and the sensations on the other hand show that determinateness in a qualitative or intensive scale carries with it the same logical consequences as does determinateness of magnitude for physically measurable quantities. Furthermore determinateness in either case is only approximately attainable, whether we rely upon the immediate judgments of perception or are able to utilize instruments of measurement. The practical impossibility of literally determinate characterisation must be contrasted with the universally adopted postulate that the characters of things which we can only characterise more or less indeterminately, are, in actual fact, absolutely determinate<sup>1</sup>.

<sup>1</sup> The notion of the Determinable will be shown in later chapters to have importance in a large number of applications.



## CHAPTER XII

## THE RELATION OF IDENTITY

§ 1. THE occasion for using the relation of identity is where a common term appears in different connections; thus we use the idea of identity always along with the idea of difference. The logical relation between difference—or more properly otherness—and identity, is that of co-opponency: that is, taking *A* and *B* as any two terms, it cannot be that *A* is both identical with and other than *B*, and it must be that *A* is either identical with or other than *B*. Thus the relation between identity and otherness is reciprocal. It must therefore be explained that we cannot *define* otherness as meaning non-identity any more than we can *define* identity as meaning non-otherness. The conceptions of identity and of otherness must be separately and independently understood before we can assert the above axioms.

The most trivial and apparently insignificant use of the relation of identity is expressed in the formula '*x* is-identical-with *x*,' where what is primarily meant is that in repeated occurrences of the word *x*, either in a special context or irrespective of context, the word shall mean in any later occurrence what it meant in an earlier occurrence. Thus, even in this very elementary usage, the idea of identity goes along with the idea of otherness; for identity applies to what is *meant* by

the word, and otherness to its several occurrences. Underlying this characteristic of language there is the corresponding characteristic of thought; thus, in using '*x* is-identical-with *x*' in reference to entities and not mere words, we return in thought to the object previously thought of, so that identity applies to what constitutes the *object* of thinking, and otherness to the several recurrent *acts* of thought.

A less elementary usage of the relation of identity occurs in the definition of words or phrases; thus, if *x* and *y* stand for two different phrases, we may speak of *x* being identical with *y*, although the phrases are palpably different. Here otherness applies to the phrases and identity to what is meant by the phrases. Verbal identification of *x* with *y* may be contrasted with factual identification; here the relation of identity applies (as before) to the objects denoted by the words; but the *proposition* asserting identity in the one case is of a different nature from the *proposition* asserting identity in the other case: for in the first case it is verbal, in the second factual. The relation of identity asserted in the two propositions: 'Courage is the mean between timidity and foolhardiness' and 'Courage is the one virtue required of a soldier' is the same: but the natures of the propositions differ, since the first—being put forward as a definition—is verbal, and the second is factual. More generally, we may distinguish the different grounds—such as rational, experiential or linguistic—upon which the assertion of any specific logical relation is based; but these differences in the grounds of assertion, do not affect the nature of the relation asserted. Thus, abbreviating 'is identical with'

into 'z,' from the verbal statement *xiz* together with the factual statement *yiz* we may correctly infer the factual statement *xiz*. This inference uses the transitive<sup>1</sup> property of identity, and would therefore be impossible unless the relation of identity asserted in a verbal statement was the same as that asserted in a factual statement.

§ 2. We have shown the proper sense in which difference can be said to be involved in identity, but many philosophers have laid down the dictum that identity implies difference, in the sense, apparently, that when we assert that *A* is identical with *B* we are also involved in the assertion that *A* is different from *B*. The plausibility of this dictum depends upon a certain looseness in the application of the word 'implies'; thus the statement that identity implies difference is correct in the sense that the *asserting* of identity between one pair of terms implies our having implicitly or tacitly *asserted* difference between *another* pair of terms. This follows from what has been said above; e.g. when identifying the colour of this with the colour of that, we are implicitly differentiating 'this' from 'that'; and thus the identification and the differentiation may properly be said to be component parts of a single mental act. But, to give a more precise statement of this implication, it would be necessary to say that, when *A* is identical with *B* in a certain respect, then *A* is different from *B* in some other respect. In common language it is of course perfectly legitimate to say of two things that they are identical in respect

<sup>1</sup> For the term 'transitive' as a property of the relation of identity see Part I, Chapter XIV.

of colour and different in respect of shape. But here the term 'identity' should be used more precisely: we ought properly to say that, while the colour of this is identical with the colour of that, the shape of this is different from the shape of that. When then identity and difference go together in this way we ought to say, not that the *things* are both identical and different, but that one of their *qualities* is identical and another different. To complete the account of the modes in which identity and difference mutually involve one another without confusion, we need only take four such typical elementary propositions as:

- (i) this is *p*,
- (ii) this is *q*,
- (iii) that is *p*,
- (iv) that is *q*,

where *p* and *q* stand for different qualities, and 'this' and 'that' for different things. Then, comparing (i) with (ii) or (iii) with (iv), we have identity of thing and otherness of quality; and, comparing (i) with (iii) or (ii) with (iv), we have otherness of thing and identity of quality; finally, comparing (i) with (iv) or (ii) with (iii), we have both otherness of thing and otherness of quality.

§ 3. The relation and distinction between thing and quality may be generalised in terms of the correlative notions of substantive and adjective, the latter admitting of further resolution into determinates and their determinables. Thus, when predicating the same adjective *p* of this and of that substantive, we shall say that this and that *agree* as regards the determinable

$P$ ; and when predicating  $p$  of this and  $p'$  of that substantive, we shall say that this and that *disagree* as regards the determinable  $P$ . Here the term 'disagree' is used in place of 'differ,' for, strictly speaking, 'difference' is a relation between adjectives under the same determinable; and in measuring different degrees of difference amongst such adjectives, we may speak of the substantives as being similar when the degree of difference between the adjectives characterising them is small, and as being dissimilar when the degree of difference is great. Further we may say that two substantives partially disagree when they are characterised by the same determinates under certain determinables and by different determinates under certain other determinables. But partial agreement must be distinguished from approximate agreement, otherwise called similarity; and partial disagreement must be distinguished from remote disagreement, otherwise called dissimilarity. The distinction between similarity and dissimilarity involves reference to adjectives under the same determinable, and is obviously a matter of degree; while partial agreement or disagreement involves reference to adjectives under different determinables.

§ 4. Adjectives under the same determinable are usually said to be comparable; whereas those under different determinables are said to be incomparable or disparate. This point raises a question of considerable psychological interest as to the possibility of comparing two such disparate characters (say) as red and a trumpet-blast. The possibility of such comparison may perhaps be accounted for by association; or it may be that some

real deep-rooted form of connection underlies the two characters, which, if it could be explicitly rendered, would have its logical place amongst relations such as those which we are now discussing. But apart from this possible topic of psychological interest, it is not usual to speak of 'red' and 'a trumpet-blast' as being either like or unlike; and it is more usual to restrict the use of the terms *like* and *unlike* to qualities belonging to the same determinable, such as colour or sound. If this is admitted, the conclusion at once follows that like and unlike are not proper logical contradictories, but are relations of degree; so that to say of two comparable qualities that they are more or less like is equivalent to saying that they are less or more unlike; we cannot define the point at which the relation of difference changes from likeness to unlikeness, their opposition being only one of degree. When we compare different degrees of difference between determinates under a determinable whose variations are continuous, and judge, for instance, that the difference between  $A$  and  $C$  is greater than that between  $A$  and  $B$ , such differences between the determinates may be said to have distensive magnitude. When this distensive magnitude is too small, we fail perceptually to discriminate (say) between  $A$  and  $B$ ; and some psychologists have virtually taken 'identity' to be equivalent, in such cases, to minimum discernible difference. This, however, entails logical contradiction; for the conception of a 'minimum discernible difference' implies that we fail to discriminate between qualities, which really *are* different and not identical; and that strict identity can only be predicated when difference has reached the

*absolute* limit zero. On this ground, it might be urged that identity when applied to qualities susceptible of continuous variation *means* zero difference; and has, therefore, a different significance from identity, when applied to things or to qualities which vary discretely. This contention would, in my view, be fallacious; for it would appear to involve a confusion between the objective conception of identity itself and the subjective limitations in our power of judging identity. On the other hand, *difference* when applied to adjectives under the *same* determinable has a certain meaning which is distinct from any meaning of difference applicable to substantives or to adjectives under one and another determinable. As regards the latter, difference can only mean mere *otherness*; but as regards the former, difference may mean more than mere otherness; viz. something that can be measured as greater or smaller. Thus Socrates is merely other than Plato, red is merely other than hard; but round and square, red and yellow, five and nine are not merely non-identical, but are also such that the difference between them can be apprehended as greater or smaller (say) than that between oblong and square, orange and yellow, seven and nine.

§ 5. There is yet another aspect of the dictum: identity (of adjective) implies difference (of substantive) according to which it could equally well be rendered: difference (of adjective) implies difference (of substantive). For, where identity applies to the adjective and difference to the substantive, identity may properly be said to imply difference, in the sense that the identity predicated of an adjective is used along

with otherness as predicated of compared substantives; but, in this sense, we may also say that difference implies difference; i.e. that difference predicated between two adjectives is used along with otherness as predicated of the compared substantives. The dictum should therefore be expressed in more general terms to include identity and difference in respect of the adjectives characterising one and another substantive. Thus: Comparison with respect to any determinable character, whether it yields identity or difference, presupposes otherness of the substantives characterised by the determinable in question. In this connection we may examine the contrast commonly drawn between *qualitative difference* and *numerical difference*. This terminology is incorrect, for 'numerical difference' simply means *otherness*—the very notion 'numerical' owing its origin to the conception of mere otherness, which is the basis of number. Again in contrasting qualitative with numerical difference there is the suggestion that otherness does not apply to qualities or adjectives, whereas in its developments into number otherness is clearly seen to apply precisely in the same way to adjectives as to substantives. In our view the required distinction is that which was drawn above between the word difference as meaning merely otherness, and the word difference in its exclusive application to adjectives under the same determinable.

§ 6. Under the head of difference and otherness a special problem to be discussed is that involved in the famous Leibnizian principle 'the identity of indiscernibles.' This phrase signifies that 'indiscernibility implies identity,' which is an awkward way of saying that

or the 'dissimilarity of the diverse' -  
see Mr. Suggart's  
'The Nature of Good'

'otherness implies discernibility.' Here the term *discernibility* has not a psychological but a purely ontological significance. More explicitly, the phrase signifies that a plurality of existent objects is only possible so far as there is some difference in the qualities or relations which can be predicated of them. If, in this phrase, the term relation is interpreted to include such external relations as space or time, then no reasonable criticism of the Leibnizian formula could be maintained; for, as has been contended from another point of view in Chapter II, 'existential otherness' implies difference in spatial or temporal relations. But this interpretation can hardly be taken to represent Leibniz's meaning, since he denied external relations, and held that this denial demonstrates the non-reality of space. But whereas he pretends to base the denial of space upon his dictum, in reality his dictum would have no plausibility unless it had been previously agreed that space was unreal. What Leibniz certainly meant was that *two* existent objects could not agree in all their *internal* characters and relations. The difficulty that here arises in regard to the number of respects and the remoteness of difference that are abstractly necessary for the possibility of two-ness of existence exhibits more emphatically the purely dogmatic nature of the Leibnizian principle, which seems to me in any case to have no logical justification whatever<sup>1</sup>.

<sup>1</sup> Much the same considerations were brought forward in the criticism of Bradley's dictum that 'distinction implies difference' (see Chapter II). If the Leibnizian and the Bradleyan principles can be in any way distinguished, it is that the former is ontological and the latter epistemological.

§ 7. Having discussed the notion of identity in its contrasts and connections with difference and otherness, we must finally examine the nature of the relation of identity itself, apart from its connections with other relations. The problem may be indicated by discussing whether identity is or is not definable. For this purpose it will be desirable to begin by considering the formula *xix*, rather than *xiy*. For *xiy* can only be interpreted by explicitly distinguishing the word, phrase or symbol which denotes an entity from the entity itself that is denoted; and moreover the proposition *xiy* (until definite equivalents for *x* and *y* are substituted) can only be adopted hypothetically or illustratively. On the other hand, the straightforward proposition *xix* is to be *asserted* on rational grounds for *any* value of *x*. Eliminating the symbol *x*, what is to be universally asserted is that 'Any entity is identical with itself.' Unfortunately the term *itself* can only be defined as 'what is identical with *it*'; and hence any explication of the formula seems to lead to an infinite regress. This difficulty can be removed by expressing the formula in the negative form: 'No entity is *identical* with any entity *other than* itself.' This is to be understood as a brief way of asserting: 'No entity (*x*) is *identical with* any entity *other than* what is *identical with x*.' This axiom expresses a universal and rationally grounded truth, expressed in terms of the two relations, identity and otherness. I shall attempt to show that the conceptions of identity and of otherness are two independent indefinables, the understanding of which is required in order intelligently to accept the truth of the above axiom or of any other proposition which explicitly or implicitly



involves the notions of identity or of otherness. Now the above axiom of identity (*xix*) is never explicitly used anywhere but in an abstract logical or philosophical context; on the other hand *xiy* is explicitly used in concrete logical and mathematical formulae; and its usage in such cases always involves the process of *substituting* *y* for *x*. The connection between identity and substitution is roughly expressed in the rule: Given *xiy*, we may always substitute *y* for *x*. More exactly, taking *p* to be any predication, 'If *x* is identical with *y*, then *x* is *p* would imply *y* is *p*.' The converse of this is also generally admitted: viz. that 'If for every value of *p*, *x* is *p* would imply *y* is *p*, then *x* must be identical with *y*.' From these two assertions conjoined it follows that the proposition '*x* is identical with *y*' is equipollent or co-implicant with the proposition that 'For every predication *p*, *x* is *p* would imply *y* is *p*.' The problem thus arises whether this equipollence or co-implication can serve as a *definition* of identity. My ground for rejecting such a view is that the equipollence asserted could not be understood or utilised unless we understood what is meant by the assertions '*x* is identical with *x*,' '*y* is identical with *y*' and '*p* is identical with *p*,' and had accepted these assertions as true, because of our independent understanding of what is meant by the relation of identity.

In the converse form of the identity formula we have admitted that if all the adjectives that characterise a substantive *x* also characterise a substantive *y*, then *x* and *y* are identical. But the hypothesis here is really impossible, for the adjective 'other than *y*' cannot characterise *y*. Hence there is one adjective at least

that must characterise *x* but not *y*. What then is the one relational adjective that must characterise any one existent and not any other? It must be existence itself. For to exist means to *stand out from amongst other things*. Otherness is thus presupposed by existence. In short, if the existent is what is manifested in time and space, and if time and space are wholes divisible into parts, then the only necessarily differentiating mark of one existent must be its temporal and spatial position. This brings us back to the Leibnizian formula, but at the expense of admitting the reality of time and space as the condition of otherness—a condition which is both necessary and sufficient.

x  
'Existence'  
presupposes  
'Otherness'

Although in the sense explained identity always implies the legitimacy of substitution, we cannot say conversely that the legitimacy of substitution always implies identity. For whenever two predications are *co-implicative*, the one may always be substituted for the other in the same way as for substantives which are *identical*. Thus, for substantives *x* and *y*, we have the formula:

If *x* is identical with *y*, then '*x* is *p*' is co-implicative with '*y* is *p*,' where *p* is any predication applicable to *x* and *y*.

Corresponding to this, for predications *q* and *r*, we have:

If *q* is co-implicative with *r*, then '*q* is *n*' is co-implicative with '*r* is *n*,' where *n* is any predication applicable to *q* and *r*.

Thus, given the co-implication of the two predications (*q*) human being, and (*r*) featherless biped, we can infer that the proposition 'the number of human beings is *n*'

*Johnson's view appears to be that 'existence' means 'having spatial & temporal position'—i.e. that the relation is a logical one—so that it would be a contradiction in terms to suppose that a particular exists which is not in space & time. This view seems to leave little room to be abused! We do*



is co-implicative with the proposition 'the number of featherless bipeds is  $n$ .' And again given the co-implication of the two propositions ( $q$ ) 'There is a righteous God' and ( $r$ ) 'The wicked will be punished,' we can infer that the proposition 'That there is a righteous God is problematic' is co-implicative with the proposition 'That the wicked will be punished is problematic.' These examples show that the relation of co-implication between predications has some of the same properties as that of identification between substantives, and therefore co-implication is apt to be conceived (and even by some symbolic logicians has actually been symbolised) as equivalent to identification. It appears to me that it is theoretically possible that the conception of co-implication could be shown to correspond to factual identification; but this indeed is doubtful, because the relation of co-implication is compound, i.e. it denotes the conjunction of the two correlatives implying and implied by, whereas it seems impossible to reduce the notion of identification to the conjunction of two correlatives.

Before dismissing this subject, it must be admitted that both as regards substitution for identified substantive terms, and substitution for co-implicative predications, certain limitations seem to be required. For example Mr Russell has familiarised us with illustrations for the necessity of this limitation by such examples as that, from the identification of 'Scott' with 'the author of *Waverley*,' we cannot, by substituting the one term for the other in such a proposition as 'George IV believed Scott to have written *Marmion*' infer that 'George IV believed the author of *Waverley* to have written *Marmion*.' Or again that the number of the

apostles is identical with the number of months in the year, does not necessarily imply that anyone who doubts that the number of apostles is 12, would necessarily doubt that the number of months was 12. It appears that the only statements for which such substitutions are invalid are secondary propositions predicating of a primary proposition some or other relation to a thinker. X § 8. To complete the account of identity, it is necessary to anticipate what will be elaborated in a later part of the work; and for this purpose the discussion will be restricted to identity of substantives proper, excluding any further reference to the identity of adjectives or predications. A substantive proper I have defined as what is manifested in space and time, or otherwise an existent; and the category existent has been divided into the two sub-categories called respectively occurrent and continuant. Now identity as applied to an occurrent could be illustrated thus: "The flash of lightning to which I am pointing is identical with the flash of lightning to which you are pointing." A continuant, on the other hand, means that which continues to exist while its states or relations may be changing; identity of continuant may therefore be illustrated by some such examples as 'The body which illuminates the earth is identical with the body that warms the earth'; 'The person who was experiencing the tooth-ache is identical with the person who intends to go to the dentist.' This last conception we find discussed at some length under the heading 'Identity' by the earlier English writers Locke, Hume and Reid, who used the term to signify personal identity instead of giving to it the merely relational significance of the more modern conception. In

See Moore's *Logic*  
on "Psychology"  
(pp. 217 ff.) for  
discussion of  
section.

effect, for them the assertion or denial of identity is equivalent to the assertion or denial that there is a person who continues to exist throughout a period of time in which his various experiences may be altering in character. The destructive view represented by Hume regarded experiences as what I call occurents, any one of which is merely replaced by another in the course of time. This destructive view is equivalent to the denial of any psychical continuant. Necessarily an occurrent experience is as such identical with itself; what Hume denied was that kind of connection between one such occurrent and another which constitutes them into alterable states of one individual self. Hence it was not the general conception of identity, but the reality of the psychical continuant that Hume denied and that Reid maintained. When this old problem is revived at the present day, it is usually formulated as the question whether there is any ultimate philosophical justification for regarding one and another experience as belonging to the same self. This phraseology, however, is misleading; for it appears to assume the existence of a self, and to raise only the question as to whether we can refer different experiences to the same self; whereas the real problem is whether there is a self at all. In discussions, by Hume and others, connected with that of 'personal identity,' we find the same problem raised as to the validity of the more general notion 'substance'—i.e. in my terminology, a continuant (whether psychical or physical). Now we might interpret Kant as admitting the validity of the conception of a physical continuant while denying that of a psychical continuant. On the other hand, Berkeley more obviously supported

Hume

a.

b.

f.

1.

2.

the notion of a psychical continuant and rejected that of a physical continuant. At the present day, most philosophers who reject these notions regard them as reached by a constructive process, and therefore as being merely convenient fictions. To assume that a notion is necessarily fictitious because it owes its origin to a constructive process is fallacious, especially for those who accept the central Kantian position according to which the objective validity of any conception is established by showing it to be the result of a synthetic, i.e. constructive act of pure thought. The same relation between identity and the notion of a continuant applies to the physical as to the psychical continuant. In the physical sphere, those who reject the physical continuant maintain that any physical event is a mere occurrent which is replaced by another occurrent; and that a so-called change is merely the temporal succession of one and another occurrent. Those, on the other hand, who accept the notion of a physical continuant maintain the validity of the notion of change as distinct from mere alternation, and as therefore presupposing the conception of a physical continuant.

We may indicate more positively the distinction between the two views which respectively reject and accept the notion of a continuant, while agreeing in the application of the relation of identity. Those who deny continuance of existence as well as those who affirm it can legitimately collect all the experiences, occurring during some part or the whole of what we call an individual's total experience, to constitute a class and assert that this collection is identical with itself. On the other hand, for those who affirm the continuant, the collection is

Logical construct  
(cf. Russell)Hume  
(cf. Mill)

a.

? What does all  
this mean?

B

not a mere plurality but a specific kind of 'unity'; in other words, they hold that an intimate bond of causality subsists between the experiences attributable to one individual of a kind which does not subsist between experiences arbitrarily selected from the histories of different individuals. The notion of this unique kind of bond is, on this view, the product of a constructive act, but not to be dismissed on this ground as merely fictitious.

## CHAPTER XIII

## RELATIONS OR TRANSITIVE ADJECTIVES

§ 1. So far we have treated the adjective solely in its reference to the substantive which it characterises. We have now to consider a type of adjective whose meaning when analysed exhibits a reference to some substantive other than that which it characterises. Thus we may characterise a certain child by the adjective 'liking a certain book,' or a certain book by the adjective 'pleasing a certain child.' These adjectives predicated respectively of the child and of the book; are complex; and when we take the substantival reference out of this complex, there remains the term 'liking' or 'pleasing.' Such terms do not function as completed adjectives, and will be called relational adjectives. Propositions involving adjectives of this type may be ranged in a series according to the number of substantives to which they refer. Thus, in the following examples: '*A* is wise,' '*A* likes *B*,' '*A* gives *X* to *B*,' '*A* accuses *B* at time *T* of *C*,' the number of substantival references are respectively one, two, three and four, and the corresponding adjectives or propositions may be called monadic, diadic, triadic and tetradic.

Taking first the two-termed relation, let us consider the proposition '*X* likes *Y*' or '*X* is greater than *Y*.' The notion of '*X* as liking *Y*' or of '*X* as being greater than *Y*' is to be distinguished from the notion of '*Y* as liking *X*' or '*Y* as being greater than

*See Intro. § 2.*

*X*.' At the same time the thought of any assigned relation of *X* to *Y* involves the thought of a definitely assignable relation of *Y* to *X*; for example, the thought of *X* as liking *Y* involves the thought of *Y* as pleasing *X*; and *X* as greater than *Y* involves *Y* as being less than *X*. Two relational adjectives such as liking and pleasing, or greater than and less than—each of which in this way involves the other—are called correlatives, and either one is said to be the converse of the other. When the relation is expressed by a transitive verb, the opposition between active and passive expresses the mutual implication of correlatives: thus, '*X* likes *Y*' means '*Y* is liked by *X*,' or '*Y* pleases *X*' means '*X* is pleased by *Y*.'<sup>1</sup> Except in the case of the active and passive voice, there is no general rule of language according to which the converse of a given relative can be expressed, and therefore a special knowledge of words in current use is required in order to be able to express a relation in its converse form; as when we pass from '*X* is greater than *Y*' to '*Y* is less than *X*,' or from '*X* likes *Y*' to '*Y* pleases *X*.' However, the fact expressed in terms of a relative is the same as the fact expressed in terms of its converse, whether the terms employed are philologically cognate or not.

It must be pointed out that 'liking *Y*' or 'liking someone,' etc., is a completed adjective; and, in general, out of a relational adjective we may construct a com-

<sup>1</sup> Comparing '*x* sleeps' or '*x* is sleeping' with '*x* hits *y*' or '*x* is hitting *y*,' and noting that '*sleeps*' is an intransitive and '*hits*' a transitive verb, we ought properly to call *sleeping* an intransitive and *hitting* a transitive adjective. Thus a *relation* is properly defined as a '*transitive adjective*,' the ordinary adjective being distinguished as *intransitive*.

pleted adjective by supplementing the substantival reference. And conversely most ordinary adjectives in use can be analysed so as to elicit a relational element as a component. For instance 'amiable' contains the relational element 'liked by,' and may be roughly defined 'liked by most people.' Again, substantive words are constructed out of relational adjectives, e.g. 'a shepherd' which means 'a person who takes care of sheep.' It is noteworthy, however, that to take the substantives 'shepherd' and 'sheep' as examples of correlatives involves a double error, since the true correlatives involved in the meaning of shepherd are 'taking care of' and 'taken care of by,' which are adjectival and not substantival; while the meaning of the word 'sheep' contains no relational element at all.

§ 2. Our immediate concern will be with diadic adjectives, otherwise called coupling. Given any two substantives—say *x* and *y*—we may construct what will be termed a substantive-couple expressed by the phrase '*x* to *y*,' which is to be distinguished from the substantive couple '*y* to *x*.' Similarly, given any two correlative coupling adjectives—say greater-than and less-than—we construct what will be termed an adjective-couple, expressed by the phrase 'greater-than to less-than.' The significance of a substantive-couple is to be explained by defining it as that which may be characterised by an adjective-couple; and the significance of an adjective-couple, by defining it as that which may characterise a substantive-couple. Thus the relation of substantive-couple to adjective-couple is the same as that of an ordinary adjective to an ordinary substantive; and just as the latter are united through the

characterising tie, so are the former. Again, just as the extension determined by an ordinary adjective comprises the substantives of which the adjective may be truly predicated, so we may say of an adjective-couple that the extension which it determines comprises the substantive-couples of which the adjective-couple may be truly predicated. This relation between the substantive-couple and the adjective-couple is brought out by expressing the proposition ' $x$  is greater than  $y$ ' in the form

' $x$  to  $y$  is-as greater than to less than,'

and the proposition ' $y$  is less than  $x$ ' in the form

' $y$  to  $x$  is-as less than to greater than.'

This mode of formulation helps perhaps to explain the process of relational conversion, which may be illustrated as follows:

- (1)  $x$  is greater than  $y$ ,
- $\therefore$  (2)  $x$  to  $y$  is-as greater than to less than,
- $\therefore$  (3)  $y$  to  $x$  is-as less than to greater than,
- $\therefore$  (4)  $y$  is less than  $x$ .

In passing from (1) to (2), the introduction of the term 'less than' depends merely upon knowledge of the arbitrary usage of language; but the logical validity of the step rests upon the fundamental principle of thought that every relation has its converse. Each step also requires that the order in which the adjective terms are mentioned is to be understood to correspond to that in which the substantive terms are mentioned. Similar reformulations could be applied to triadic and

higher orders of adjectives: thus ' $x$  receives  $b$  from  $y$ ' will be rendered

$x : b : y$  | is-as | receiving : given by : giving to ;

and ' $y$  gives  $b$  to  $x$ ' becomes

$y : b : x$  | is-as | giving : given to : receiving from ;

where ':' stands for 'to,' and the ordering of the words is to be interpreted cyclically.

§ 3. The general notion of an adjective-couple that can be predicated of a substantive-couple is familiarly illustrated in what is called analogy. Take the proposition 'England to Australia is-as parent to child'; here the predicate is what I call an adjective-couple, constituted by taking the relation 'parent-of' and its converse 'child-of'; while the subject is a substantive-couple composed of the two substantives England and Australia. The copula 'is-as' marks a statement of analogy. Another example with the same adjective-couple is 'France to Algiers is-as parent to child.' From these two predications of the same adjective-couple we should infer that 'France is to Algiers as England is to Australia.' This form of proposition, however, differs importantly from that which predicates an adjective-couple of a substantive-couple. For in affirming the equivalence of the relation in which France stands to Algiers with that in which England stands to Australia, there is no indication of the kind of relation in respect of which the two substantive-couples agree. We might, in fact, compare this inference with the inference that ' $X$  is like  $Y$ ' which could be drawn from the two propositions that ' $X$  is red' and ' $Y$  is red'; or equally from the two propositions that ' $X$  is square' and ' $Y$  is

square.' We then see that such terms as similar or analogous when used to connect two substantives or two substantive-couples are quite indeterminate with respect to the ground of similarity or analogy; so that from the assertions '*A* is similar to *B*' and '*B* is similar to *C*' it could never be inferred that '*A* is similar to *C*.' My formulation of the relational proposition '*A* to *B* is-as *p* to *q*' is of course suggested by the arithmetical expression for a ratio and the equality of ratios, but here there is a danger of misunderstanding; for in my phraseology we could assert that '8 to 6 is-as greater by 2 to less by 2,' and again that '5 to 3 is-as greater by 2 to less by 2' and hence that '8 to 6 is-as 5 to 3': but in arithmetic the phraseology '*p* to *q*' is understood to denote exclusively a relation under the genus called *ratio*, and could not be applied to a relation under any other genus such as difference or distance, etc., etc. Instead of calling such a term as ratio, difference or distance, etc., by the familiar name *genus*, it ought, properly speaking, to be termed 'relational determinable' in contradistinction to three-fifths, or minus two, or three yards distant, which are 'relational determinates' under their respective relational determinables. Whereas *here* the phraseology '*a* to *b* is-as *p* to *q*' is used for relational determinates under *any* relational determinable, in arithmetic this phraseology is limited to relational determinates under the one relational determinable called *ratio*. It must be observed that, when the term analogy is explicitly referred to a given relational determinable, such as *ratio*, then, from the assertion that two substantive couples are analogous to the same substantive couple, we may infer that the

two are analogous to one another. This is precisely parallel to the case of agreement or similarity when explicitly referred to any given adjectival determinable, such as colour.

§ 4. This account of relational adjectives leads to a consideration of a species of tie distinct from the characterising tie, which we shall call the *coupling tie*. In the phrase '*x* to *y*' the word *to* has been chosen to indicate this tie, and hence the effect of the coupling tie is to construct a substantive-couple. Any preposition or prepositional phrase such as *of*, *by*, *for*, *at*, *with*, *in*, *in reference to*, indicates the presence of the coupling tie. We must not, however, in general say that the preposition denotes merely a tie; for a difference of preposition often stands for a difference in the relation predicated. For example '*x* is influenced to move *towards y*' has a different meaning from '*x* is influenced to move *away from y*'; where the difference of preposition is seen to entail a difference of relation—namely the difference between attraction and repulsion. In fact, prepositions used along with adjectives or verbs express determinate *modifications* of relation. The essential feature of a tie, on the other hand, is that it is incapable of modification, and hence we frequently find that it does not enter as a separate verbal component in a sentence.

Whenever a tie (whether it be the characterising tie, or the coupling tie, or any other) does not appear as an actual word, there are conventions of language which indicate its presence. In languages in which inflexion is largely used, such as Latin and German, there are two main kinds of grammatical rule; namely, the rules



of concordance and the rules of governance. We shall find that the rules of concordance point to the presence of the characterising tie; and those of governance to the coupling tie. The rules of concordance are, briefly, that adjectives and verbs must agree in gender, number and case, with the substantives that they characterise; so that the characterising tie is not necessarily expressed by use of the verb 'to be' but merely by inflexion. On the other hand, the rules of governance always determine the case—genitive, dative, accusative, or ablative—of the substantive that is introduced along with any transitive verb, relational adjective, or preposition. We find, especially in Latin, that considerable changes in the *order* of words (which may vary for purposes of rhetorical significance) are permissible because of the inflexions which are understood to indicate (i) by grammatical *agreement*, how the words are to be attached in thought by the characterising tie, and (ii) by grammatical *governance*, how they are to be attached in thought by the coupling tie. Furthermore, where modification of case occurs (with or without a preposition), not only the coupling tie, but also the special modification of relation that is to be understood is grammatically indicated. The characteristic of English, in contrast to highly inflexional languages, is that no inflexions are required by rules either of concordance or of governance, except in the two instances: (i) for differences of person and number in many verbs (which illustrate the *characterising* tie), and (ii) the accusatives—him, her, me, us, them, whom (which illustrate the *coupling* tie). All the other instances of inflexion in English—for example the possessive pronouns and the

tenses of verbs—are used, not according to any rules of concordance or governance, but to express distinctions of meaning. The difference between the two kinds of inflexion—the one being *significant* and the other *syntactic*—is brought out by comparing the English 'her father' or 'his mother,' where the difference of gender is significant and not syntactic, with the French 'son père' or 'sa mère,' where the difference of gender is syntactic and not significant. In English, the conventional rules of concordance or governance are replaced—except in the two cases mentioned above—by the equally conventional *ordering* of the words.

§5. The coupling tie—which might have been called the prepositional tie, in consideration of the grammatical rules of governance, or again the relational tie, in consideration of the philosophical problems that have been raised in regard to the nature of relation—is of fundamental importance in discussing one of the paradoxes that Mr Bradley and others have found in the general notion of relation. The paradox is briefly brought out in the following contention: when we think of  $x$  as being  $r$  to  $y$ , we have first to relate  $x$  to  $y$  by the relation  $r$ , and then relate the relation  $r$  to  $x$  by—say  $r'$ —and  $r$  to  $y$  by—say  $r''$ , another relation. This again will require that  $x$  should be related to  $r'$  by the further relation  $r'''$ , which will lead to an infinite regress on the side of  $x$ , and a similar regress on the side of  $y$ . This paradoxical contention is met by pointing out that in constructing an object out of the constituents  $x$ ,  $r$ , and  $y$ , we do not introduce another constituent by the mere act of constituting these constituents into a unity. The pretence of paradox is due to the assumption that to

the act of relating or constructing there corresponds a special *mode* of relation; so that a tie is confused with a relation. That a tie and a relation are distinct is brought out by considering the fact that if, for a given adjective—whether ordinary or relational—we substitute another adjective, we shall have constructed a *different* unity; but, if we drop the characterising tie with a view to replacing it by some adjective or relation, then either the unity itself is destroyed, or it will be found that the characterising tie remains along with the adjective or relation so introduced. Similarly, the coupling of terms is not a *mode of relating* them for which another mode of relation could be substituted; for, if they were uncoupled, again the unity would be destroyed.

The distinction between a tie and a relation may be brought out from another point of view by the consideration that the specific difference between one kind of tie and another is determined by the logical nature of the constituents tied. Thus the use of an adjective in general involves the characterising tie, by which it is attached to a substantive; and the use of a relational adjective in particular further involves the coupling tie by which the two substantive-terms are attached to one another. On the other hand, where terms are related by a genuine relation, their logical nature allows any specific relation to be replaced by any other, this other being in general under the same relational determinable.

Now just as we must distinguish between any relation and the relational (or coupling) tie, so I have throughout assumed, and it is of the first importance to emphasise, the distinction between characterisation and

the characterising tie. I have, in fact, spoken repeatedly of the *relation* of adjective to substantive; and this is the relation called characterisation—a specific kind of relation to be distinguished, for instance, from 'liking' or 'exceeding,' etc. While characterisation, then, is a relation, the characterising tie (like any other tie) is a mode of *connection*, represented usually by the participle 'being.' This participle may be expanded into 'being-characterised-as-being,' and it will still represent merely the characterising tie. Thus the following series of examples are seen to be but different modes of expressing precisely the same fact: I am thinking (1) of this |as| tall; (2) of this |as-being| tall; (3) of this |as-being-characterised-as-being| tall; where the phrase enclosed in vertical lines represents nothing more nor less than the characterising tie. The equivalence between 'S as-being P' and 'S as-being *characterised* as-being P,' etc., is precisely analogous to the arithmetical equalities: ' $S \times P = S \times 1 \times P = S \times 1 \times 1 \times P$ ,' where S stands for any magnitude and P for any numerical multiplier, while the number *one* takes the place of the relation '*characterised*,' and the operator ' $\times$ ' takes the place of the tie 'as-being.' We will therefore call 'characterisation' the unit relation, because it may be conceived as a factor in every adjective and in every relation. The conception of 'characterised' as a relational factor, analogous to *one* in the expression  $S \times (1 \times P)$ , will be shown more precisely by adopting a different mode of bracketing (3) whereby it becomes: (4) I am thinking of 'this |as-being| characterised-as-being-tall.' Now this last expression is formally analogous to (5) I am thinking of this |as-being| taller-than-that. The phrase

The distinction between a 'relation' and a 'characterising tie' is important, but none of Johnson's explanations seem to make it clear. We may note that in a relational prop. - A & B - the relation is a component of the prop.; whereas in a substantive-adjective prop. - it is not - the characterising tie is.

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|as-being|, occurring both in (4) and (5), represents the characterising tie; but in (5) the relation (with its tie) is expressed by 'taller-than,' and supplemented by the substantive-term 'that,' while in (4) the relation (with its tie) is expressed by 'characterised-as-being' and supplemented by the adjective-term 'tall.' Since, moreover, we may expand the tie in (5) into 'as-being-characterised-as-being,' we see that 'characterised' is a latent factor in every relation, as well as in every ordinary adjective. Thus 'characterised' is a latent factor even in the relation 'characterising': for 'I am thinking of *P* as characterising *S*' may be expanded into 'I am thinking of *P* |as-being| characterised as characterising *S*.' This reformulation incidentally shows that, though *characterised* has the properties of a unit relation, yet its converse *characterising* has not these properties.

§ 6. Now the form of proposition in which 'characterised' is introduced explicitly as a relation, derives its significance and its legitimacy from our having taken an adjective—namely 'tall'—as a term. We are therefore extending the application of the notion of a relation, when in this way we take an adjective as *term*, instead of (as hitherto) a substantive. Indeed no limit can be imposed upon the kind or category of entity which may constitute a 'term' of which adjectives or relations to other entities may be predicated. In particular, we must recognise that certain adjectives may be significantly predicated of adjectives and of propositions, and even of relations; and that certain relations may be significantly predicated as subsisting between a substantive and an adjective, or between one adjective and another, or between one proposition and another, or even between

one relation and another. The adjectives, relations, or propositions of which other adjectives or relations may be predicated must when so connected be called *terms*, in contrast with the adjectives or relations predicated of them. The logical mode in which adjectives, relations or propositions enter as *terms* into a construct, is reflected in language by the substantival form assumed by them; e.g. intolerance, hatred, the enthusiasm of the people, that matter exists, to be or not to be, etc. etc.

Underlying the merely nominal question whether adjectives, relations or propositions when functioning as substantives in a construct should be *called* terms, two philosophical questions arise, which I shall here deal with rather summarily. First, is it literally the same *entity* which can be treated indifferently *either* as an adjective in its primary or natural functioning, *or* as a quasi-substantive of which certain other adjectives or relations may be predicated? To this I give an affirmative answer; and the objections to my view are (I think) met by insisting that the adjectives or relations which may be significantly predicated of *primary* adjectives or relations (as they may here be called) belong to a different logical sub-category from these latter, and may be called *secondary*. Thus, it is not that the primary adjective changes its category when functioning as quasi-substantive, but it is that the secondary adjective must be said to belong to a special sub-category, determined by the category of the primary adjective of which it may be predicated. But a second question arises: whether the relation or adjective that is *apparently* predicated of a relation, adjective or proposition, is *really* so predicated; or whether it is pre-

*This is answered  
cf. his definition  
of substantives:  
This definition  
given adjectives  
would be mistaken  
& not merely  
substantives.*

II.

licated rather of certain substantives to which there is implicit or explicit reference. As regards this question, it must be pointed out that while relations, subsisting primarily between certain entities entail relations between other entities involved in or connected with the former, yet the relations thus entailed are not identical with the primary relation; so that whether a relation is predicated of this or of that kind of entity needs separate discussion in each type of case. A typical example will be discussed in a succeeding paragraph.

§ 7. We now propose to analyse relational propositions of more complicated forms. The principles (previously illustrated) in accordance with which a diadic proposition may be reduced to a monadic by hyphenating (or bracketing) the predicate, and may be converted by substituting for the given relation its correlative, may be extended to relational propositions of higher orders. In such propositions any one of the substantive-terms may be taken as subject, and a complex containing the remaining substantive-terms as monadic predicate; or any couple of substantive-terms may be taken as subject and a complex containing the remaining substantive-terms as diadic predicate; and so on. In such transformations, any *permutations* that are made amongst the substantive-terms, will require the substitution of 'cognate' modes of expressing the given relation; and, as in diadic relations, the order of relationality is reduced by constructing a complex predication indicated by a hyphen or bracket. For example, the triadic proposition 'A gives X to B' is reduced to a diadic by transforming it into 'A gives-X to B' or 'B is receiver-from-A of X' or 'X is given-to-B by A,' which again may be converted,

respectively, into 'B receives-X from A,' 'X is given-by-A to B,' 'A is giver-to-B of X.' Thus the single radical relation represented by such a verb as 'to give,' from which a triadic proposition is constructed, gives rise to three pairs of converse forms (making six correlatives), namely: giver-to and receiver-from; receiver-of and given-to; given-by and giver-of. And, furthermore, each of the above six diadics becomes monadic by completing the hyphening of the predicate without any further verbal alteration.

In the above illustrations a *single* radical relation was involved; but another kind of complication arises when, besides a number of *substantival* components, the proposition contains more than one *adjectival* or *relational* component, these being in general indicated by different verbs. For example, the proposition 'A prevented B from hurting C' contains the three substantive-terms A, B, C, together with two verbs, of which 'prevent' is the principal and 'hurt' is sub-ordinate. Such a proposition, treated merely as triadic, may be transformed by permutation into (e.g.) 'B was prevented by A from hurting C,' and by bracketing into 'C was saved-from-being-hurt-by-B by A.' But another form of analysis may be used where—as in the case before us—we are dealing, not only with a plurality of substantive terms, but also with a plurality of radically different relational components. In this analysis, we take as a bracketed constituent a complex containing all the components of a complete proposition; viz. the proposition that contains the *subordinate* relation or adjective. A proposition in this aspect may be called a *possibile*. The *possibile* in our illustration is 'B's-hurting C,' and

the proposition predicates of the person *A* the relation 'preventing.' The distinction between the two modes of bracketing—in both of which a triadic proposition is reduced to a diadic—will be shown by comparing: '*A* is |preventer-from-hurting-*C*| by *B*' with '*A* |prevents| the-hurting-of-*C*-by-*B*.' In the former the *terms* of the relation are the two persons '*A*' and '*B*,' and the relation may be expressed:

'*A*' to '*B*' is-as 'preventer-of-hurt-to-*C*-from' to 'prevented-from-hurting-*C* by';

in the latter the *terms* are the person '*A*' and the *possible* '*C*'s being hurt by *B*,' the relation being:

'*A*' to '*C*'s being hurt by *B*' is-as 'preventer-of' to 'prevented-by.'

The immediate purpose of this illustration is to show that a proposition that predicates a relation of a term to a *possible*, entails also the predication of certain complex relations between this principal term and the various substantive constituents of the *possible*. The example under consideration may be more fully expressed by explicit reference to the action of *A* which was causally preventive: thus, '*A*'s-instructing-*D* prevented *B*'s hurting-*C*.' This more explicit form of statement is typical of the causal proposition as such which, properly speaking, always relates one *possible* to another either by way of production or of prevention. For example, such a statement as 'The earth causes the fall of the stone' is an elliptical expression for the more fully analysed proposition 'The-proximity-of-the-earth causes the-fall-of-the-stone.' Since, however (as follows from the above remarks), a diadic relation of one *pos-*

*sible* to another entails certain more complex relations amongst the substantive constituents of the *possibilia*, it is formally legitimate to assert that the earth stands in *some* relation to the fall of the stone, or again that the earth stands in *some* relation to the *stone*, though each of these relations is more complex than the diadic relation of causation, which holds between the two *possibilia*.

§ 8. The above discussion leads to the special problem of the nature of the relation of assertion. Consider the simplest case: '*A* asserts *S* to be characterised by *P*.' This contains the three terms *A*, *S* and *P*, together with the two verbs or relations 'assert' and 'characterise,' of which 'assert' is the principal, and 'characterise' the subordinate. Taking the complex as expressing a triadic relation of *A* to *S* to *P*; it may be reformulated in several ways: such as, '*A* asserts-*P*-to-characterise *S*'; '*P* is asserted-by-*A*-to-characterise *S*,' which predicate diadic relations of *A* to *S*, and of *P* to *S* respectively. But the more natural mode of expressing the relation is as one of the thinker *A* to the *possible* '*S* being (characterised by) *P*.' My account of the relation of causation as holding primarily between *possibilia* but also as entailing relations (of a higher order than diadic) amongst the component terms of the *possibilia*, must be applied to any such relation as that of interrogating, doubting, considering, affirming, denying, etc., in which a thinker may stand to a *possible* or *assertibile*. The relation of the thinker to the proposition as a whole does not preclude—it rather *entails*—relations to the constituents of the proposition. Thus, the relation of the thinker to the subject *S* is expressible in terms of *P*,



viz., 'asserts-*P*-to-characterise'; and that of the thinker to the predicate *P* is expressible in terms of *S*, viz., 'asserts-*S*-to-be-characterised-by.'

This principle—that assertion or judgment involves a relation of the thinker to each of the constituents of the proposition as well as to the proposition as a unitary whole—is familiarly (but misleadingly) expressed in some such form as that 'in judgment the thinker asserts a relation of one idea to another.' If by idea was really meant 'the object of a thought' rather than 'the thought of an object,' this statement would, in my view, be essentially correct. Or more briefly: I agree that judgment relates the object of one thought to the object of another thought; but I deny that judgment relates the thought of one object to the thought of another. That is to say, though judgment requires us to think about objects, the judgment is about these objects, and not about our thinking about them. The account of judgment that is to be rejected involves a confusion between a primary proposition which is about objects, and a secondary proposition (as it may here be termed) about our ideas of objects; and, if this identification of a primary with a secondary proposition is consistently carried out, we should have to interpret a secondary proposition as a tertiary; namely, as being about our ideas about our ideas of objects, and so *ad infinitum*. But admitting, as I have done, that in judgment we assert a relation of one object of thought to another, say of *S* to *P*, it is necessary further to consider what sort of *idea* we have of *S* or of *P* when we judge that '*S* is *P*'; and here I propose to distinguish between the specific and the generic idea of *S* or of *P*. The *specific* idea of *S* that is involved

in doubting or asserting the proposition, is the idea of *S* as 'characterised by *P*'; and this idea includes the *generic* idea of *S* as 'characterisable,' i.e. as of the nature of a substantive. Similarly the *specific* idea of *P* is the idea of *P* as 'characterising *S*' and this includes the *generic* idea of *P* as 'characterising'; i.e. as of the nature of an adjective. But here it is to be observed that the *relation* that may be said to be predicated, viz. that of characterisation, does not subsist between the idea of *S* and the idea of *P*, since each of these ideas is specifically completed in the single complex idea of '*S*-as-characterised-by-*P*' or of '*P*-as-characterising-*S*' or again of 'the-characterisation-of-*S*-by-*P*'; and I hold that these three phrases express different modes of *constructing* one and the same construct or complex object of thought. No difference of principle is involved when dealing with an explicitly material relational proposition, such as '*S* is  $\hat{R}$  to *T*.' Here the *specific* idea of *S* is the completed idea of *S* as 'being  $\hat{R}$  to *T*'; that of *T* is the completed idea of *T* as 'being  $\check{R}$  to *S*'; that of  $\hat{R}$ , as 'relating *S* to *T*'; and that of  $\check{R}$  as 'relating *T* to *S*.' These *specific* ideas include such *generic* ideas as that of *S* and of *T* as being substantives and of  $\hat{R}$  and  $\check{R}$  as being a pair of relations correlative to one another. As in our simpler illustration, the completed idea is of a complex object of thought constructed out of three constituents (*S*, *R*, *T*) bound together in a certain form of unity.

\* cf. Russell's theory of judgment as a multiple relation.



## CHAPTER XIV

## LAWS OF THOUGHT

§ 1. It has been customary to apply the phrase Laws of Thought to three specific formulae; but the application of the phrase should be extended to cover all first principles of Logic. By first principles we mean certain propositions whose truth is guaranteed by pure reason. It is often too hastily said that logic as such is not concerned with truth but only with consistency; as if a conclusion were guaranteed by formal logic merely because it is consistent with any arbitrarily assumed premisses. But this entirely misrepresents the function of formal logic, which is not permissive, but rather prohibitive. It guarantees the truth—not of any proposition that is consistent with the premisses—but only of the proposition whose contradictory is inconsistent with the premisses. And even this statement goes too far; for logic does not allow any arbitrarily chosen premisses to be taken as true; and thus the only conclusions that it can be said in any sense to guarantee are those which have been correctly inferred from premisses that are themselves true. When consistency is placed in a kind of antithesis to truth, it seems often to be assumed that logic is indifferent to truth. That the reverse is the case is shown by the consideration that to say that a conclusion is validly drawn from given premisses is tantamount to asserting the truth of a certain composite proposition, viz. that the premisses imply the conclusion.

In enunciating and formulating the fundamental principles of Logic, we shall not enter into the question whether they are all independent of one another, nor into the problem as to how a selection containing the smallest possible number could be made amongst them from which the remainder could be formally derived. This problem is perhaps of purely technical interest, and the attempt at its solution presents a fundamental, if not insuperable, difficulty: namely, that the procedure of deriving new formulae from those which have been put forward as to be accepted without demonstration, is governed implicitly by just those fundamental logical principles which it is our aim to formulate explicitly. We can, therefore, have no assurance that, in explicitly deriving formulae from an enumerated set of first principles, we are not surreptitiously using the very same formulae that we profess to derive. If this objection cannot be removed, then the supposition that the whole logical system is based on a few enumerable first principles falls to the ground.

§ 2. The charge has been brought against all the fundamental principles of Formal Logic that they are trivial; or otherwise that they are nothing but tautisms. Now a tautism may be defined as a proposition which is (1) true, and (2) accepted by everybody on mere inspection as true; and these are just the characteristics required of a fundamental principle of logic. Hence to charge the fundamental formulae with being mere tautisms is not to condemn them, but to admit that they are fitted to fulfil the function for which they are intended. This function is to enable us to demonstrate further formulae, some of which, though true, are not accepted by every-

body on mere inspection as true. It is an actual fact that by means of truisms and truisms alone we can demonstrate truths which are not truisms. The above and similar criticisms directed against the fundamental formulae of Logic will be best met by directly examining this or that formula so as to bring out its precise significance in view of the different points of view from which it has been criticised; and we shall adopt this plan as occasion offers.

§ 3. Before enunciating the fundamental principles in detail, we will enquire into what is implied in speaking of them as 'Laws.' The word *law* is closely connected with the notion of an imperative; and many logicians of the present day hold that the so-called laws of thought are no more imperatives than are the axioms of arithmetic or geometry. With this view I agree, inasmuch as the axioms of mathematics can themselves be regarded as having an imperative aspect; but this is because all truth may be so regarded. The idea of truth and falsity, in my view, carries with it the notion of an imperative, namely of acceptance and rejection—a corollary from the theory which insists on the reference of judgment and assertion to the thinker. For it is only so far as assertion is recognised to be a mental act, that the notion of an imperative becomes relevant. An imperative of reason implies a restraint upon the voluntary act of assertion—a restraint which does not, however, infringe the freedom that characterises every volition, since the obligation to think in accordance with truth is self-imposed. Any study of which imperatives constitute the subject-matter has been called a *normative* science, and normative sciences have

been contrasted with positive sciences. But from a certain point of view every science may be said to exercise an imperative function in so far as any mistake or confusion in the judgments of the ordinary man is corrected or criticised by the scientist as such. Every science therefore can without any confusion of thought be regarded as normative; which is only another way of saying that the notions of truth and falsity as predicable of propositions carry with them the notions 'to be accepted' or 'to be rejected' understood as imperatives. But an explanation can be given for the restricted use of the term normative to logic, aesthetics and ethics: viz., that, while each deals with a certain kind of mental fact, it does not deal with it merely *as* fact. Every science which deals with man, either in his individual or social capacity, takes as its topic the description of mental facts—including an analysis of how men think, feel and act; but such a descriptive study of our thoughts, feelings and actions (including their causal relations) treated generally, historically or speculatively, is to be distinguished from the study of precisely the same facts in relation to certain norms or standards, and from the critical examination of these norms or standards themselves. The division of *sciences in general* into normative and positive is, therefore, unsound, inasmuch as *all* sciences may be regarded as normative in the sense that they are potentially corrective of mistaken, false or obscure views. This division (into normative and positive) is therefore properly restricted to sciences dealing with psychological material; thus the positive or descriptive treatment of mind—in its thinking, feeling or acting aspect—is (like all sciences) normative in the sense of *a.*

being potentially corrective of false judgments on the topics directly dealt with; while the treatment in Logic, Aesthetics and Ethics of these same processes is normative in the more special sense that these sciences examine and criticise the norms of thought, feeling or action themselves. Within the range for which the antithesis between normative and positive holds, the distinction between a descriptive or causal account of psychological or sociological matters, and an examination of standards or norms, is now-a-days of the first importance, inasmuch as the substitution of *causal description* in the place of *evaluation of standard* has been woefully common in works which profess to found Ethics upon psychology or sociology.

§ 4. To return to the consideration of the principles which exercise an imperative function. The fundamental formulae for conjunctive and composite propositions have been given in the chapter on compound propositions; these must be included in the general consideration of the Laws of Thought. Certain of these laws, and in particular the Reiterative, Commutative and Associative laws of Conjunction are—not only the materials which explicitly compose the logical system—but are also implicitly used in the process of building up the system. Thus, for example, we not only explicitly formulate the Reiterative Law, but in making repeated use of this or of any other law, we are implicitly using the Reiterative Principle itself. This will be seen to hold in the same way of the Commutative and Associative principles of Conjunction. Finally, inasmuch as the system is developed by means of inference, the essential principles of implication are not only *explicitly*

formulated in the formulae for composite propositions, but also *implicitly* used in constructing the logical system itself.

§ 5. The next set of laws to be considered will be those which express the nature of *Identity*, since this is a formal conception which applies with absolute universality to all possible objects of thought whatever the category to which they may belong. Identity is a relation, and as such has certain properties which are exhibited in what we shall call the *Laws of Identity*.<sup>v.</sup> Relations in general may be classified according to the formal properties they possess, irrespectively of the terms related. It will be necessary here to introduce and define three of these properties, viz., transitivity, symmetry and reflexivity. Using the symbols  $x, y, z$ , to stand for the terms of any relation, and the symbols  $\hat{r}$  and  $\check{r}$  for any relation and its converse, then,

(1) the relation  $\hat{r}$  is called transitive: when ' $x$  is  $\hat{r}$  to  $y$ ' and ' $y$  is  $\hat{r}$  to  $z$ ' together implies ' $x$  is  $\hat{r}$  to  $z$ ' for all cases of  $x, y, z$ ; for example, *ancestor, greater than, causing, implying*;

(2) the relation  $\hat{r}$  is called symmetrical: when ' $x$  is  $\hat{r}$  to  $y$ ' implies ' $y$  is  $\hat{r}$  to  $x$ '; in other words, when ' $x$  is  $\hat{r}$  to  $y$ ' implies ' $x$  is  $\check{r}$  to  $y$ ,' for all cases of  $x$  and  $y$ ; for example, *cousin, incompatible with, other than*;

(3) the relation  $\hat{r}$  is called reflexive: when ' $x$  is  $\hat{r}$  to  $x$ ' for all cases of  $x$ ; for example, *compatriot of, simultaneous with, homogeneous with*.

Now the three Laws of Identity are most simply expressible by the statement that identity is (1) transitive, (2) symmetrical, (3) reflexive; or otherwise, for

every object of thought (represented by the symbols  $x, y, z$ ).

- (1) *Transitive Law*: If  $x$  is identical with  $y$ , and  $y$  is identical with  $z$ ; then  $x$  is identical with  $z$ .
- (2) *Symmetrical Law*: If  $x$  is identical with  $y$ , then  $y$  is identical with  $x$ .
- (3) *Reflexive Law*:  $x$  is identical with  $x$ .

It will be observed that there are a host of other relations which have these same three properties; e.g. contemporaneous, homogeneous, compatriot, numerically equal, equal in magnitude, etc., but analysis of every such relation shows it to contain a reference to some identical element, upon which these formal properties depend.

§ 6. The phrase 'Law of Identity' has been traditionally used for one of the three fundamental logical principles, known as the Laws of Identity, of Non-Contradiction, and of Excluded Middle, to which the term 'Laws of Thought' has been usually restricted; but, since these three laws relate exclusively to *propositions*, whereas the conception of identity applies to all objects of thought, I propose to substitute for the traditional terminology, the Principles 'of Implication,' 'of Disjunction' and 'of Alternation' respectively; and to insert a fourth, to be called the 'Principle of Counter-implication.' The four together will be entitled 'the Principles of Propositional Determination.' The four laws are thus brought into line with the four forms of composite proposition discussed in a preceding chapter. The composite propositions expressed in their general form, i.e. in terms of two *independent* components  $p, q$ , are of course not guaranteed as true by pure logic; in

other words, they require material or experiential certification as opposed to merely formal or rational certification. The *principles*, on the other hand, are those cases of the composite propositions, expressed in their quite general form, the truth of which is guaranteed by pure logic. For the purposes of formulating the principles on the lines of the four composite functions, we may slightly modify the expression of these latter as follows:

- |                                   |                                     |
|-----------------------------------|-------------------------------------|
| (1) Implicative Function :        | If $P$ is true, then $Q$ is true.   |
| (2) Counterimplicative Function : | If $P$ is false, then $Q$ is false. |
| (3) Disjunctive Function :        | Not both $P$ true and $Q$ true.     |
| (4) Alternative Function :        | Either $P$ true or $Q$ true.        |

The *principles* are obtained by substituting  $P$  for  $Q$  in the implicative and counterimplicative functions, and  $P$ -false for  $Q$ -true in the disjunctive and alternative functions. Thus:

### *Principles of Propositional Determination*

( $P$  being any proposition)

- |                                 |                                                     |                     |
|---------------------------------|-----------------------------------------------------|---------------------|
| (1) <i>Implicative</i> :        | It must be that if $P$ is true, then $P$ is true,   | (Identity)          |
| (2) <i>Counterimplicative</i> : | It must be that if $P$ is false, then $P$ is false, |                     |
| (3) <i>Disjunctive</i> :        | $P$ cannot be both true and false,                  | (non-contradiction) |
| (4) <i>Alternative</i> :        | $P$ must be either true or false,                   | (Excluded Middle)   |

where the words 'must be' and 'cannot be' serve to indicate that the principles are formally or rationally certified.

This formulation uses a single proposition  $P$  together with the two adjectives true and false, in preference to the more usual mode of expression which employs two propositions,  $P$  and not- $P$ , and a single adjective 'true'; as in the following:

- (1) If  $P$  is true, then  $P$  is true.
- (2) If not- $P$  is true, then not- $P$  is true.
- (3)  $P$  and not- $P$  cannot both be true.
- (4) Either  $P$  or not- $P$  must be true.

There are several reasons for adopting the former of these two modes of formulation in preference to the latter. In the first place it uses the comparatively simple notion of  $P$  being false instead of the rather awkward notion of not- $P$  being true. Secondly it enables us to define 'contradiction' by means of the principles, which would be impossible without a circle if we introduced the contradictories  $P$  and not- $P$  into the formulation. In the third place, the introduction of the phrases  $P$ -true and  $P$ -false is in accordance with the fact that the adjectives true and false are the first characteristics by which the nature of the proposition as such is to be understood. A closer analysis of this formulation of the alternative and disjunctive principles will throw further light on the nature of the antithesis between the adjectives true and false. We have emphasised the point that these adjectives are predicable only of propositions; in other words 'anything that is true or false is a proposition'; the principle of alternation adds to this statement its complementary, viz., 'anything that is a proposition is true or false.' It is clear, of course, that these two statements are not synonymous. Again, the principle of disjunction states that the adjectives true and false are incompatible; and this again goes beyond what is explicitly involved in the statement that they are predicable exclusively of propositions.

The most obvious immediate application of these

principles is obtained by taking  $P$  to stand for a definite *singular* proposition: ' $s$  is  $p$ ,' where ' $s$ ' stands for a uniquely determined or singular subject, and ' $p$ ' for any adjective. Then  $P$ -false becomes ' $s$  is not- $p$ .' In this application, the four principles may be called the Principles of Adjectival Determination, and assume the following form:

|                                  |                                             |
|----------------------------------|---------------------------------------------|
| Principle of Implication:        | If $s$ is $p$ , then $s$ is $p$ .           |
| Principle of Counterimplication: | If $s$ is not- $p$ , then $s$ is not- $p$ . |
| Principle of Disjunction:        | $s$ cannot be both $p$ and not- $p$ .       |
| Principle of Alternation:        | $s$ must be either $p$ or not- $p$ .        |

In this application, the principles are expressed in terms of any adjectives  $p$  and not- $p$  predicated of any subject  $s$ ; instead of being expressed in terms of the adjectives true and false predicated of any proposition  $P$ . In ordinary logical text-books the 'Laws of Thought' are almost always expressed in this specialised form; but, by this mode of enunciation, the *generality* which characterises the formulation in terms of propositions is lost; for when 'adjectives predicated of any subject' is substituted for 'propositions,' we have only a special case from which the general could not have been derived. It is convenient for many purposes to use the term 'predication' to stand for 'adjective' or 'proposition'; thus we may include both the general and the special formulae of determination in the abbreviated forms 'If  $p$  then  $p$ '; 'If not- $p$  then not- $p$ '; 'Not-both  $p$  and not- $p$ '; 'Either  $p$  or not- $p$ '; where  $p$  stands for any predication. The two sets of formulae might again be expressed—without any modification of meaning—in the form of universals, since  $P$  stands for *any* proposition, and  $s$  for *any* subject; thus:



| Generalised Form for Propositional Determination | Generalised Form for Adjectival Determination |
|--------------------------------------------------|-----------------------------------------------|
| If any proposition is true, it is true           | If anything is $p$ , it is $p$                |
| If any proposition is false, it is false         | If anything is not- $p$ , it is not- $p$      |
| No proposition can be both true and false        | Nothing can be both $p$ and not- $p$          |
| Any proposition must be either true or false     | Anything must be either $p$ or not- $p$       |

A comparison between these two generalised formulations of the principles will bring out the important distinction between false and not-true and again between true and not-false. According to the principles of adjectival determination 'Anything must be either true or *not-true*'; whereas of propositions we can say 'Any proposition must be either true or false.' Now, since it is only propositions of which truth is properly predicable, therefore of anything that is not a proposition the adjective true must be denied; thus we must say 'The table is not true' on the elementary ground that 'the table' is not a proposition; but we cannot say that 'The table is false,' because it is only propositions which can be said to be false. Thus the principles of propositional determination force upon us the notable consideration that the word false does not really mean the same as not-true. To have expressed the principle of alternation in the form 'Anything must be either true or false' without the necessary restriction to a proposition would have been actually wrong. On the other hand, the form 'Any proposition must be either true or not-true' is not sufficiently determinate, for this alternative would hold of any subject whatever and fails to express the alternative peculiar to the proposition itself.

But is equiv-  
alent to?

On the same ground, the disjunctive principle is not properly expressed in the form 'No proposition can be both true and not-true.' This affords another, and, in my view, the most important justification for formulating the principles in terms of the adjectives true and false instead of in terms of the propositions  $P$  and not- $P$ . In illustration of the above discussion we may point to the analogy between the four adjectives true, false, not-true, not-false and the four adjectives male, female, not-male, not-female. The antithesis between male and not-male or again between female and not-female is applicable to any subject whatever, but that between male and female is applicable exclusively to organisms. Analogously the antithesis between true and not-true or again between false and not-false is applicable to any subject-term whatever, but that between true and false is applicable exclusively to propositions.

cf. 'moral', 'immoral'  
and 'general', 'et al'

§ 7. Now the so-called Law of Identity—which I have expressed in the form 'If  $P$  is true then  $P$  is true' where  $P$  stands illustratively for any proposition—has been the favourite object of attack by critics of the principles of formal logic, on the score of its insignificance or triviality. The reason why the formula appears to have little or no significance is that its implicans is literally identical with its implicate. It will be found, however, that the necessary condition for the explicit use of the relation of identity is that the identified element should have entered into different contexts. It must be noted that this necessary reference to difference of context does not render the relation of identity other than absolute, i.e. it in no way implies that in its two occurrences the identified element is partly identical



and partly different. The application of this general condition to the Principle of Implication requires us to contemplate the proposition ' $P$  is true' as one that may have been asserted in different connections or on different occasions or by different persons. Then, since the formula 'If  $P$  is true then  $P$  is true' is to be understood as logically general, its full import can be expressed in the form: 'If the asserting of  $P$  in any one context is true, then the asserting of  $P$  in any context whatever is true.' If this analysis be accepted, it will be found that the principle could not have been enunciated except for the possibility of identifying an assertum or proposition as distinct from the various attitudes (belief, interrogation, doubt, denial) which might have been adopted towards it on different occasions by the same or different persons. One important element of meaning, therefore, implicit in the formula is that it tacitly implies the identifiability of a proposition as such.

Turning now to the adjective 'true' as it occurs in our analysis of the formula, let us contrast it with certain adjectives that are predicable of things in general. The principle—that what can be asserted in one context as true must be asserted in any other context as true—is more familiarly particularised in the form 'any proposition that is once true is always true'; that is to say that 'true' as predicable of any proposition is unalterable; whereas there are certain adjectives and relations predicable of things in general which may characterise them only temporarily. Contrasting, for instance, the Principle of Propositional Determination 'If any proposition is true it is true' with the Principle of Adjectival Determination 'If anything is  $p$  it is  $p$ ,'

we find that in the former the copula 'is' is to be interpreted without reference to the present or any other assigned time; whereas in the latter the adjective  $p$  may be alterable, so that the copula 'is' must here be understood as referring to definitely assigned time. In the case of anything that is at an assigned moment of time  $p$ , the principles of logic do not entitle us to assert that it will be or has always been  $p$ . Taking as examples 'The water has a temperature of  $30^{\circ}$  C.' or 'Mr B. is at home,' we must say on the one hand that if these propositions are true at any time, they are true at all times. But we must not say that if the predicate 'having a temperature of  $30^{\circ}$  C.' or the relation 'being at home' is true of a given subject at one time, it will be true at all times. This obvious comment would not have been required if language had distinguished in the mode of the verb 'to be' between a timeless predication and a tense (present, past or future). Certain logicians have, however, deliberately denied the dictum that what is once true is always true, and their denial appears to be due to a confusion between the time at which an assertion is made, and the time to which an assertion refers; or as Mr Bosanquet has neatly put it—'between the time of predication and the time in predication.' Others, i.e. the Pragmatists, have made the denial of this dictum a fundamental factor in their philosophy, inasmuch as they have taken the term 'true' to be virtually equivalent to 'accepted,' whereas everybody else would agree that the term is equivalent rather to the phrase 'to be accepted.' Again, the dictum would not have been confidently admitted in the days before the principles of Logic had been formulated by Aristotle, when the

*cf. Keynes' Logic*

antithesis between the immutability of truth and the mutability of things appears to have presented an insurmountable problem. Since, then, it has been disputed from three different points of view that the truth or falsity of a proposition is independent of the time of assertion, the first Law of Thought—my interpretation of which brings out clearly this quality of truth—is effectively freed from the charge of triviality.

But not only must we interpret the principles as implying the unalterability of truth, for further, according to the principle of disjunction, a proposition cannot be both true and false; and this is to be interpreted to imply that, if a proposition is true in any one sense, there can be no sense of the word true in which it could be false, or other than true. On this interpretation the principle would be opposed by those philosophers who employ the words relative and absolute, or similar terms, to distinguish two kinds of truth. In consistency with this philosophical position, the term 'true' must be said to have two meanings, so that one and the same proposition might be true in one meaning of the term 'true' and false in another. It would seem that it is only on this theory that philosophers could maintain that a certain proposition such as 'Matter exists' is true in or for science, and at the same time false from the point of view of philosophy. According to the view, however, of those who maintain rigidly the validity of the Principles of Determination, it cannot be said that the same proposition is true in one sense and false in another sense, although it may be said, of course, that one sense given to a certain collocation of words would yield a true proposition, while another sense given to the same

collocation of words would yield a false proposition. We must deny for instance that 'Matter exists' can be true in one sense and false in another sense, though we do not for a moment dispute that 'Matter exists' in one sense may be true while 'Matter exists' in another sense may be false. It is noteworthy that the confusion here is exactly parallel to that between the time of predication and the time in predication. Thus the assertion 'Mr Brown is at home' cannot be true at one time and false at another; though that 'Mr Brown is at home at one time' may be true and that 'Mr Brown is at home at some other time' may be false.

§ 8. We propose now to consider the *Principles of Adjectival Determination* with a view to giving added significance to the predication factor by bringing out the relation of an adjective to its determinable. For this purpose the principles will be reformulated as follows:

(1) *Principle of Implication*: If  $s$  is  $p$ , where  $p$  is a comparatively determinate adjective, then there must be some determinable, say  $P$ , to which  $p$  belongs, such that  $s$  is  $P$ .

(2) *Principle of Counterimplication*: If  $s$  is  $P$ , where  $P$  is a determinable, then  $s$  must be  $p$ , where  $p$  is an absolute determinate under  $P$ .

(3) *Principle of Disjunction*:  $s$  cannot be both  $p$  and  $p'$ , where  $p$  and  $p'$  are any two different absolute determinates under  $P$ .

(4) *Principle of Alternation*:  $s$  must be either not- $P$ ; or  $p$  or  $p'$  or  $p''$ ... continuing the alternants throughout the whole range of variation of which  $P$  is susceptible— $p$ ,  $p'$ ,  $p''$ ... being comparatively determinate adjectives under  $P$ .

For convenience of reference, these formulae may be elliptically restated as follows:

- (1) If  $s$  is  $p$ , then  $s$  is  $P$ .
- (2) If  $s$  is  $P$ , then  $s$  is  $p$ .
- (3)  $s$  cannot be both  $p$  and  $p'$ , nor  $p'$  and  $p''$ , nor  $p$  and  $p''$ ....
- (4)  $s$  must be either not- $P$  or  $p$  or  $p'$  or  $p''$  or  $p'''$ ....

Contrasting this reformulation with the original formulation of the principles of adjectival determination, it will be observed that, while the predications of  $s$  are more precise, they are not so palpably obvious.

- I. The force of the first principle is that, if a subject is of such a kind that a certain determinate adjective can be predicated of it, then this presupposes that the subject belongs to a certain category such that it may be compared in character with other subjects belonging to the same category, the ground of comparison being equivalent to the determinable. The second principle states that any subject whose character is so far known that a certain determinable adjective can be predicated of it, must in fact be characterised by some absolutely determinate value of that determinable, and that, although, in many cases, such a precise determination of character is impossible, yet the postulate that *in fact* the subject has some determinate character is one that
- II. reason seems to demand. The third principle as reformulated gains in significance, as compared with the mere disjunction of  $p$  with the indeterminate not- $p$ , since now it precludes the possibility of conjoining an indefinite number of pairs of predicates, which are here exhibited as determinate and positive. In fact, in the principle of disjunction in its original form (according to which  $p$  cannot be joined with not- $p$ ) not- $p$  should

signify—not merely some or any adjective *other than*  $p$ —but some adjective that is *necessarily incompatible* with  $p$ , and the only such adjectives are those other than  $p$  which belong to the same determinable.

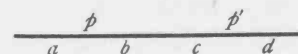
The significance of the Principle of Alternation in its new form requires special discussion. It is developed from the dichotomy 'Any subject must be either not- $P$  or  $P$ ' where  $P$  stands for any determinable. This again assumes that 'Some subjects are not  $P$ ,' i.e. that there are subjects belonging to such a category that the determinable adjective  $P$  is *not* predicable of them. The negative here must be termed a *pure* negative, in the sense that not- $P$  cannot be resolved into an alternation of positive adjectives. For example, in the statement: 'Material bodies are not conscious' the negative term 'not conscious' does not stand for any single positive determinable which would generate a series of positive determinates. We ought in fact to maintain that 'not-conscious' is not properly speaking an adjective at all; for in accordance with the reformulated Principle of Adjectival Implication, every adjective that can be predicated of a subject must be a more or less determinate value of some determinable<sup>1</sup>. Eliminating, then, the negative not- $P$  from the predicate, the reformulated Principle of Adjectival alternation may now be expressed in the form: 'Any subject that is  $P$  must be either  $p$  or  $p'$  or  $p''$  or...' where the alter-

<sup>1</sup> A negative predication of this type has sometimes been called *Privative*; but unfortunately the term privative has also been used in an opposite sense, namely, for a predication applied to a subject belonging to a category for which the positive adjective is normally applicable; as when we predicate of a person that he is *blind* or that he is (temporarily) *unconscious*.

native predication  $p$  or  $p'$  or  $p''$  or ... is restricted to subjects of which the determinable  $P$  is predicable. If we compare this form of the Principle of Alternation with the Principle of Counterimplication, viz., If  $s$  is characterised by  $P$ , it must be characterised by one or other determinate value of  $P$ , there would appear to be no obvious difference between them. The Principle of Alternation, however, supplements that of Counterimplication by implicitly postulating that the range of possible variation of the determinable can be apprehended in its completeness. The question whether we *can* in this way apprehend the complete range of possible variation of any determinable must be examined in detail. Consider the determinable 'integral number' which is always predicable of a collection or aggregate as such. Of a 'collection' we can in the first place assert universally that 'it is either zero or greater than zero,' and this it is to be observed, goes beyond the mere assertion that it is 'either zero or *not* zero.' Again we may assert that any collection is 'either zero or one or more than one,' where the alternant 'more than one' is not merely negative, but positive—though comparatively indeterminate. Proceeding in this way, we may resolve exhaustively the range of possible variations of number by an enumerated and finite series of positively indicated alternants: 'zero or one or two or three or...or  $n$  or greater than  $n$ .' What is here said of integral number holds of quantity in general, and may also be applied to any determinable (continuous or discrete) whose determinates have an order of betweenness and can therefore be serially arranged. For example, the range of hue

can be exhaustively resolved into the nine alternants 'red, or between red and yellow, or yellow, or between yellow and green, or green, or between green and blue, or blue, or between blue and violet, or violet.'

We may summarise (with some additional comments) what has been said with respect to the Principles of Adjectival Determination, formulated with reference to the determinable. (1) If  $s$  is  $p$ , then  $s$  is  $P$ . This postulates that whenever a comparatively determinate predication is asserted, then a determinable to which the determinate belongs can always be found; but it must be pointed out that language does not always supply us with a *name* for the determinable. (2) If  $s$  is  $P$ , then  $s$  is  $p$ . This postulates that in actual fact every adjective is manifested as an absolute determinate; it is to be supplemented, however, by the recognition that for a *continuously* variable determinable it is impossible actually to characterise a given subject by a precisely determinate adjective. (3)  $s$  cannot be both  $p$  and  $p'$ . This asserts that any two different determinates are incompatible; but, inasmuch as we are unable practically to characterise an object determinately (in the case of a continuously variable determinable), we must apply the formula to the case where  $p$  and  $p'$  (though only comparatively determinate) are figuratively speaking 'outside one another.' To represent this figurative analogy, suppose a point ( $a, b, c$  or  $d$ ) to represent an absolute determinate, and the segment of a line ( $p$  or  $p'$ ) to represent a comparative determinate:



then, if  $b$  is between  $a$  and  $c$ , and only then, can we assert that the (comparative) determinates  $p$  and  $p'$  are codisjunct or incompatible. (4) Any  $s$  that is  $P$  must be either  $p$  or  $p'$  or.... Here the predications  $p$ ,  $p'$ ,  $p''$ , etc. need not be absolute determinates, but to render the principle practically significant it is necessary that we should be able to compass in thought the *entire* stretch or range of variation of which  $P$  is susceptible.

§ 9. We will now pass to the principles according to which the manifested value of any one variable is determined by *its connection* with the manifested values of other variables. These principles may be expressed in forms analogous to those of adjectival determination and will be entitled the *Principles of Connectional Determination*. They embody the purely logical properties of the causal relation; but the notion of cause and effect—being properly restricted to phenomena temporally alterable—will be replaced by the wider notion of *determining* and *determined*. The characters which may be said jointly to *determine* some other character correspond to what is commonly called the cause, while any character which is thereby determined corresponds to an effect. Analysis of the general conception of causal connection reveals two complementary aspects which may be thus expressed: (a) wherever, in two instances, there is *complete agreement* as regards the cause-factors, there will be agreement as regards any effect-factor; and (b) wherever, in two instances, there is any (partial) difference as regards the cause-factors, there will be *some* difference in one or other of the effect-factors. In formulating the Principles of Con-

nectional Determination, such symbols as  $P$ ,  $Q$ ,  $R$ ,  $T$ , will be introduced to represent the characters that are connectionally *determined*, along with  $A$ ,  $B$ ,  $C$ ,  $D$ , to represent those which connectionally determine the former. Thus the conjunction  $abcd$  would correspond to a cause-complex, and  $pqrt$  to an effect-complex.

### *Principles of Connectional Determination*

(1) *Principle of Implication*. Taking any determinable  $P$ , the determinate value which it assumes in any manifestation is determined by the conjunction of a finite number of determinables  $A$ ,  $B$ ,  $C$ ,  $D$  (say), such that any manifestation that has the determinate character  $abcd$  (say) will have the determinate character  $p$  (say).

(2) *Principle of Counterimplication*. Taking any determinable  $A$ , the determinate value which it assumes in any manifestation determines (in conjunction with other factors) a conjunction of a finite number of determinables  $P$ ,  $Q$ ,  $R$ ,  $T$  (say), such that if, for instance, some manifestation having the determining character  $a$  has the determined character  $pqrt$ , then any manifestation that has the (different) character  $a'$  will have one or other of the different characters  $p'$  or  $q'$  or  $r'$  or  $t'$  (say).

(3) *Principle of Disjunction*.  $P$  being one of the characters determined by the conjunction of the determining characters  $A$ ,  $B$ ,  $C$ ,  $D$ , there can be no three instances characterised respectively by

$$abcd \sim p, a'bcd \sim p', a''bcd \sim p.$$

(4) *Principle of Alternation*. On the same hypothesis, it must be that either 'every  $abcd$  is  $p$ ' or



'every *abcd* is *p*' or 'every *abcd* is *p*'' or, etc., where the range of alternation covers all possible determinate values of *P*.

- I. The Principle of Implication postulates that the determinate value assumed by any variable is dependent in any instance not upon an indefinite number of conditions which might in some sense be exhaustive of the whole state of the universe, but upon a set of conditions that are capable of enumeration. The theoretical and practical possibility of enumerating the factors which together constitute the determining complex, enables us to express the nature of reality in universal propositions of the form 'every *abcd* is *p*.' If the character *p* could be predicated universally only of a class determined by an infinite number of conjoined characters, reality could not be described by means of universal propositions; or in other words, nature would not present uniformities which could be comprehended by thought; in short there would be nothing that could be called Laws of Nature. Hence the significance of our first principle is that reality presents uniformities that can be comprehended in thought, and that, whatever variable aspect of the universe we may be concerned with, a uniformity or law could be found such that from it the value of the variable in any manifestation could be inferred from knowledge (at least theoretically possible) of the values assumed by other variables. The Principle of Implication represents that more familiar aspect of the so-called Law of Causation expressed in terms of agreement: that in any two instances where there is complete agreement as regards the cause complex, there will be agreement

as regards the effect; or, still more colloquially, the same cause entails the same effect.

Turning now to the Principle of Counterimplication, II, this represents the other and complementary aspect of causation; namely that of difference. It postulates that we can by enumeration exhaust the characters that are determined in their variation by any cause complex; just as we assumed that the cause complex in the previous principle could be exhaustively described. In other words the effect, determined by any variation in the causal or determining complex, does not permeate the whole universe, but is restricted to some assignable sphere. This important postulate being presumed, the principle proceeds to state that, if any variable presents a different value in two instances, indications of this difference will be shown in one or other of the variables that are affected or determined by the given variable. This principle is therefore complementary to the preceding one; whereas the Principle of Implication asserts that where there is agreement in the cause there will be agreement in the effect, the Principle of Counterimplication asserts that where there is difference in the cause there will be a difference in the effect. It may perhaps even be said that in the popular conception of cause this latter aspect—viz. of difference—is more prominent than the former, viz. agreement. Here we must point out that the principles are not parallel, inasmuch as *complete agreement* in the cause is required to ensure agreement in the effect, whereas any *partial* difference in the cause will entail some difference in the effect.

A word must be said about the strictly formal relations between these Principles of Implication and Counterim-



plication. By what is familiarly known as inference by contraposition, the proposition 'Every  $abcd$  is  $p$ ' is equivalent to the proposition 'Every  $p$  is either  $a'$  or  $b'$  or  $c'$  or  $d'$ .' Similarly the proposition 'Every  $a'$  is  $p$  or  $q$  or  $r$  or  $t$ ' is equivalent to 'Every  $pqrt$  is  $a$ .' Applying this formal contraposition to the formulae for cause and effect, we see that the proposition that 'The same cause always entails the same effect' is logically equivalent to 'Any difference in the effect would entail some difference in the cause'; and again the proposition that 'Any difference in the cause will entail some difference in the effect' is logically equivalent to 'The same effect always entails the same cause.' It will be thus seen that the implicative and counterimplicative principles are not obtainable one from the other as equivalents by contraposition, but are complementary to one another, so that taken together they represent the relation between cause and effect as reciprocal. Take the one aspect of this relation; then plurality of cause holds in the sense that the effect may be partially the same in two instances where the cause is different; and plurality of effects holds in the same sense, namely, that the cause may be partially the same in two instances where the effect is different. Take the other aspect of the relation: thus, when the effect is completely and determinately characterised the character of the cause is thereby uniquely determined, just as when the cause is completely and determinately characterised the character of the effect is thereby uniquely determined. Thus, whether we are considering the relation of cause to effect or of effect to cause, the principles postulated will be in terms of complete agreement or of partial difference.

We pass now to the Disjunctive Principle. In order to expound this we must consider three instances of  $ABCD$  which agree as regards the determinate values of all but one, viz.  $A$ , of these determinables. Then, taking into consideration the Counterimplicative Principle, a difference as regards  $A$  in two instances would entail some difference in one or other of the characters that are determined by the complex  $ABCD$ . The principle then states that, if, in some pair of instances, a variation in the determining factor  $A$  entails a variation in the selected character  $P$ , then any further variation in  $A$  would entail a further variation in this same character  $P$ ; whereas if, in two instances, a variation in  $A$  entails no variation in  $P$ , then any further variation in  $A$  would entail no further variation in the same character  $P$ . It is essential to note that the Disjunctive Principle could not have been formulated as a disjunction of two types of instance, such as  $abcd \sim p$  and  $d'bcd \sim p$ . This disjunction would be equivalent to asserting that a variation in any determining factor such as  $A$  would entail a variation in *any or every* determined factor such as  $P$ ; whereas the Counterimplicative Principle has laid down only that a variation in  $A$  would entail a variation in *one or other* of the determined characters and not necessarily in every one of them. The Principle of Disjunction then supplements that of Counterimplication by maintaining that if *some one* variation in  $A$  entails a variation in the selected character  $P$ , then *any* variation in  $A$  would entail a variation in the same character  $P$ . It might be supposed, in the case where a variation of  $A$  entails no variation in  $P$ , that  $P$  is not causally connected with  $A$ , and that therefore  $A$  could be elimi-

nated. But the mere elimination of  $A$  is not in general permissible, since the character  $P$  in some one of its determinate values requires that  $A$  should be manifested in some or other of its determinate values; though, as regards the determinate value of  $P$ , it may be a matter of indifference what specific value  $A$  has. Since in this case  $A$  cannot be eliminated, it would be symbolically requisite to express the relation of determination for the case under consideration—not in the form ' $bcd$  determines  $p$ '—but in the form ' $Abcd$  determines  $p$ ,' where the significance of the symbol  $A$  is that *any* determinate value may from instance to instance be manifested without affecting the determinate value  $p$ .

IV. Fourthly, the Alternative Principle of Connectional Determination, asserts an alternation of universal propositions, and of course goes beyond any statement that could be derived from the Principle of Adjectival Alternation, in which the alternative is in the predicate. Thus the latter states the universal proposition that 'Every  $abcd$  is  $p$  or  $p'$  or  $p''$  or ...' whereas the principle under present consideration states an alternation between the universal propositions 'Every  $abcd$  is  $p$ ' or 'Every  $abcd$  is  $p'$ ' or ....

These principles will be very much more fully discussed when we deal with the topic of formal or demonstrative induction; they have been introduced at this early stage of our logical exposition in order to indicate the nature of the transition from the Principles of Propositional Determination which are purely axiomatic, to those of Adjectival Determination under a determinable, which have the character partly of axioms and partly of postulates, and from these again to the Principles of

Connectional Determination which may be taken as pure postulates.

§ 10. The formulation of the principles of connectional determination has an important bearing upon the problem of internal and external relations. In controversies on this topic it appears to be agreed that the division of relations into internal and external is both exclusive and exhaustive; and yet there seems to be no agreement as to what precisely the distinction is. One school holds that all relations are internal; the other that all are external. But on the face of it it would appear that some must be internal, others external; for otherwise it would seem impossible to give meaning to the distinction. It will be found, however, that those who deny external relations doubt, for instance, not whether spatial and temporal relations are properly to be called external, but rather whether space and time are themselves real in the sense that the real can be truly *characterised* by spatial and temporal relations; those on the other side who deny internal relations apparently hold that the independent otherness of the terms of the relation renders the relation external, inasmuch as the specific and variable relation of one term to another is not that which determines or is determined by the mere existence of the one or of the other term. The adherents then of the exclusively internal view of relations hold that the relation and its terms are mutually determinative, and the adherents of the exclusively external view, that the relation and its terms are mutually non-determinative or independent. Now it appears to me that the root misunderstanding amongst the two schools of philosophy on this point is,

not as to what is meant by an internal as contrasted with an external relation, but rather what is the nature of the terms between which the relation is supposed to subsist. The one school maintains that the relation subsists between the characters of the two related terms; the other that it subsists between the terms themselves. According to the former contention, relations are internal in the sense that they depend wholly upon the character of the terms related; according to the latter, they are external in the sense that they do not depend at all upon the mere existence of the terms qua existents. In this connection there is a further source of confusion, namely as to whether in the character of a term are to be included such relations as those of space and time, these being admittedly external, in contrast to qualities proper which are admittedly internal.

At this point I will state my solution of the problem, which will appear so simple that it would seem difficult to account for the origin of the controversy. I hold, then, that relations between adjectives as such are internal; and those between existents as such are external. In this account, adjectives are to include so-called external relations, even the characterising relation itself, as well as every other relation. The otherness which distinguishes the 'this' from the 'that' is the primary and literally the sole external relation, being itself direct and underived. And this relation is involved in every external relation. In fact, qua existent, the 'this' and the 'that' have no specific relation. The specific external relations that hold of one to another existent are derivative from their characters, in the

wider sense of character. Thus, the relation of the 'this' to the 'that' obtained from the fact that 'this is blue and that is green,' is derived from the nature of the qualities blue and green. Again the relation of proximity or remoteness obtained from the fact that 'this is here and that is there,' is derived from the positions of the 'this' and the 'that' by which their specific spatiality is characterised. The most important application of the distinction is to causal and other forms of connectional determination. Here the primary relation called cause is that between the character, dating, and locating, of two occurrences, from which the relation between the occurrences themselves is derived, the former being internal and the latter external. If there were no such internal causal relation, nothing could be stated as to the relation of event to event, except that the one is invariably accompanied by the other in a certain assignable spatial relation of space and time; and even this external relation is derived from the internal relation subsisting between the temporal and spatial *positions* occupied by the two events. If, however, all spatial properties were relative, as is maintained by Einstein and his followers, there would be no spatial relations other than internal, in fact nothing to distinguish a space from that which occupies it. The principles of connectional determination have therefore been expressed directly in terms of the characters by which the manifestations of reality may be described, from which must be derived the external relations between such manifestations themselves. It will have been observed that the correlative notions of determination and dependence enter into the formulation

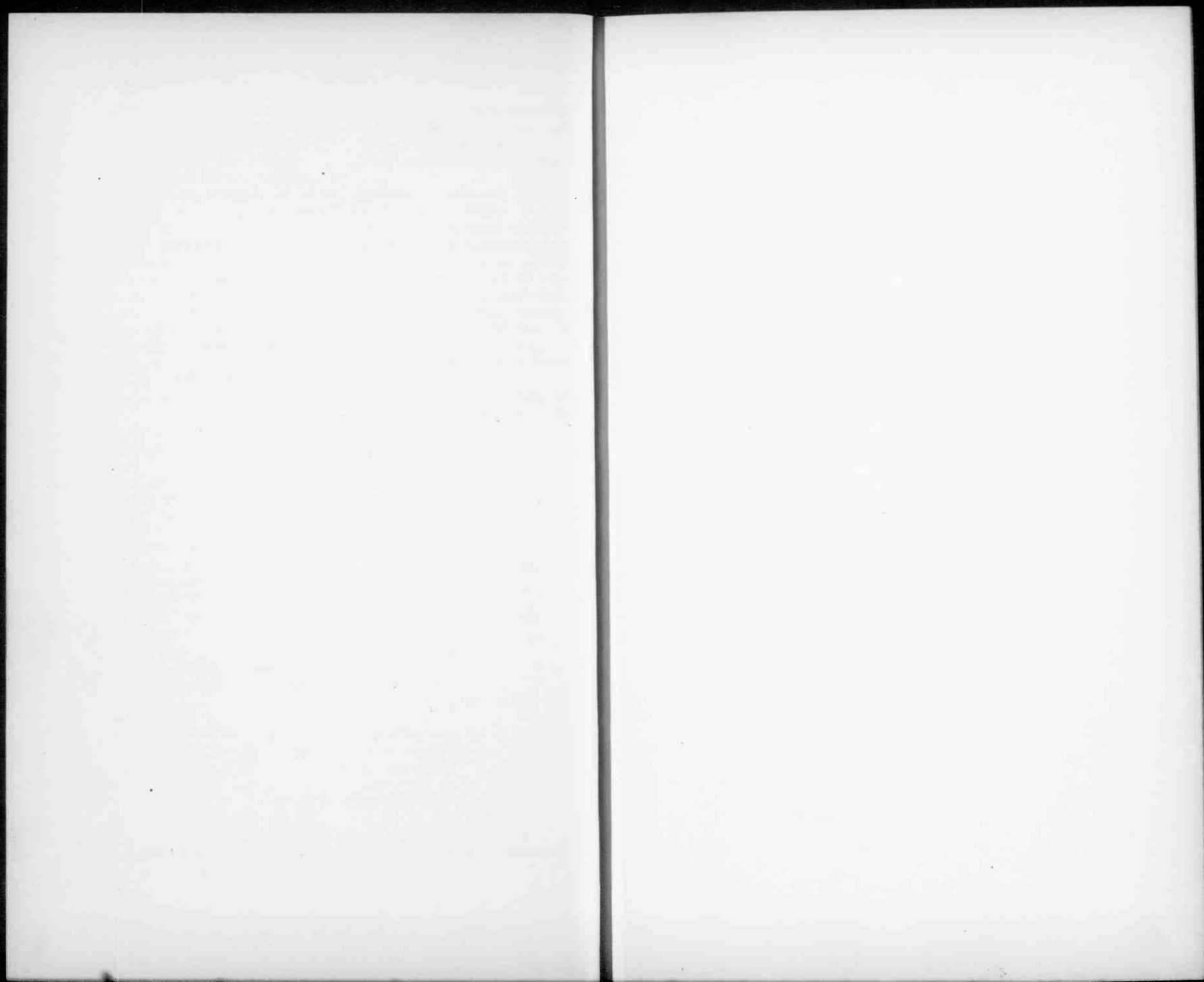
of the principles as directly applicable to the characters of manifestations and therefore only derivatively to the manifestations themselves. Hence the potential range for which these principles hold extends beyond the actually existent into the domain of the possibly existent. In this way the universality of law is wider than that of fact. While the universals of fact are implied by the universals of law, the statement of the latter has intrinsic significance not involved in that of the former.

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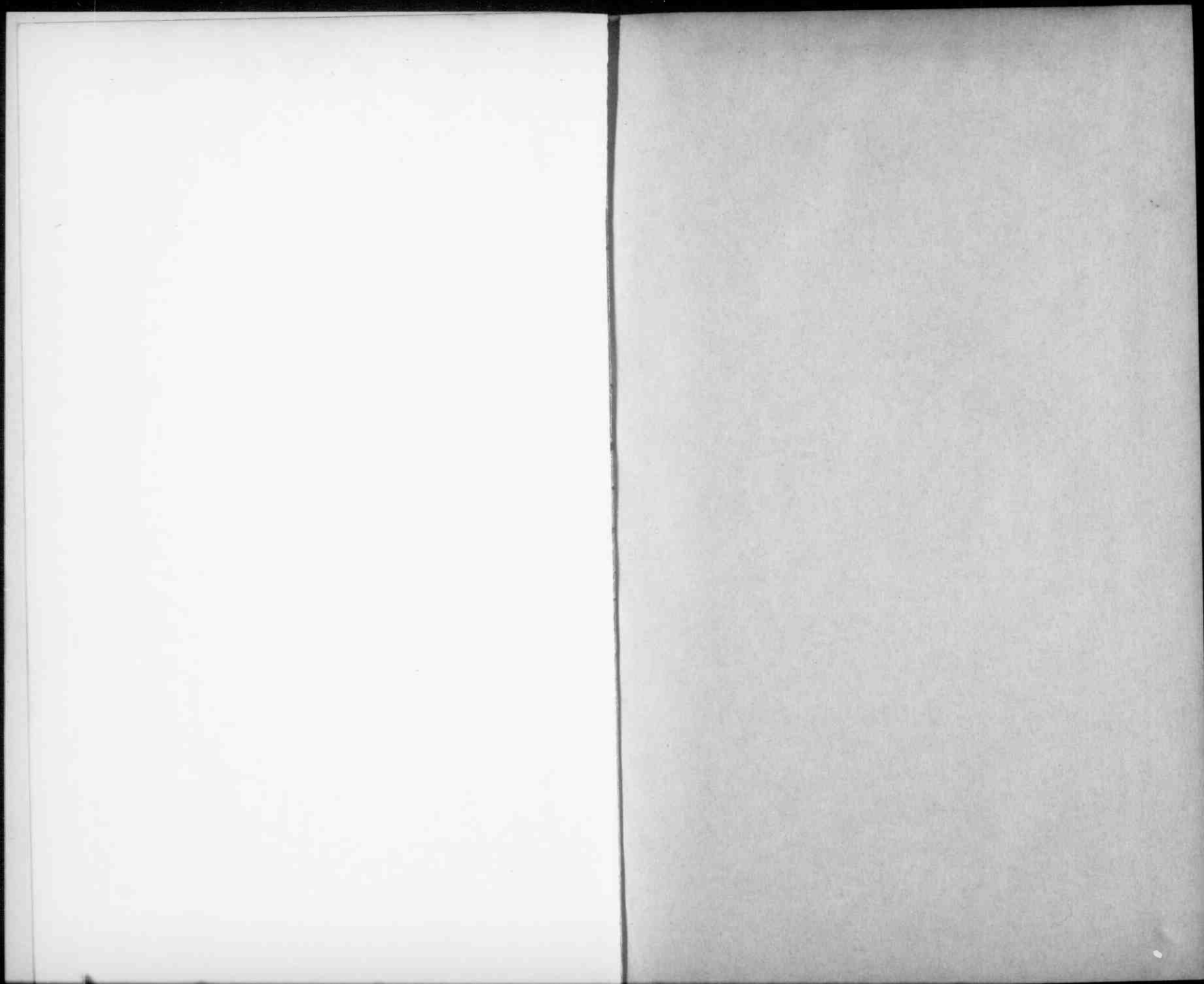
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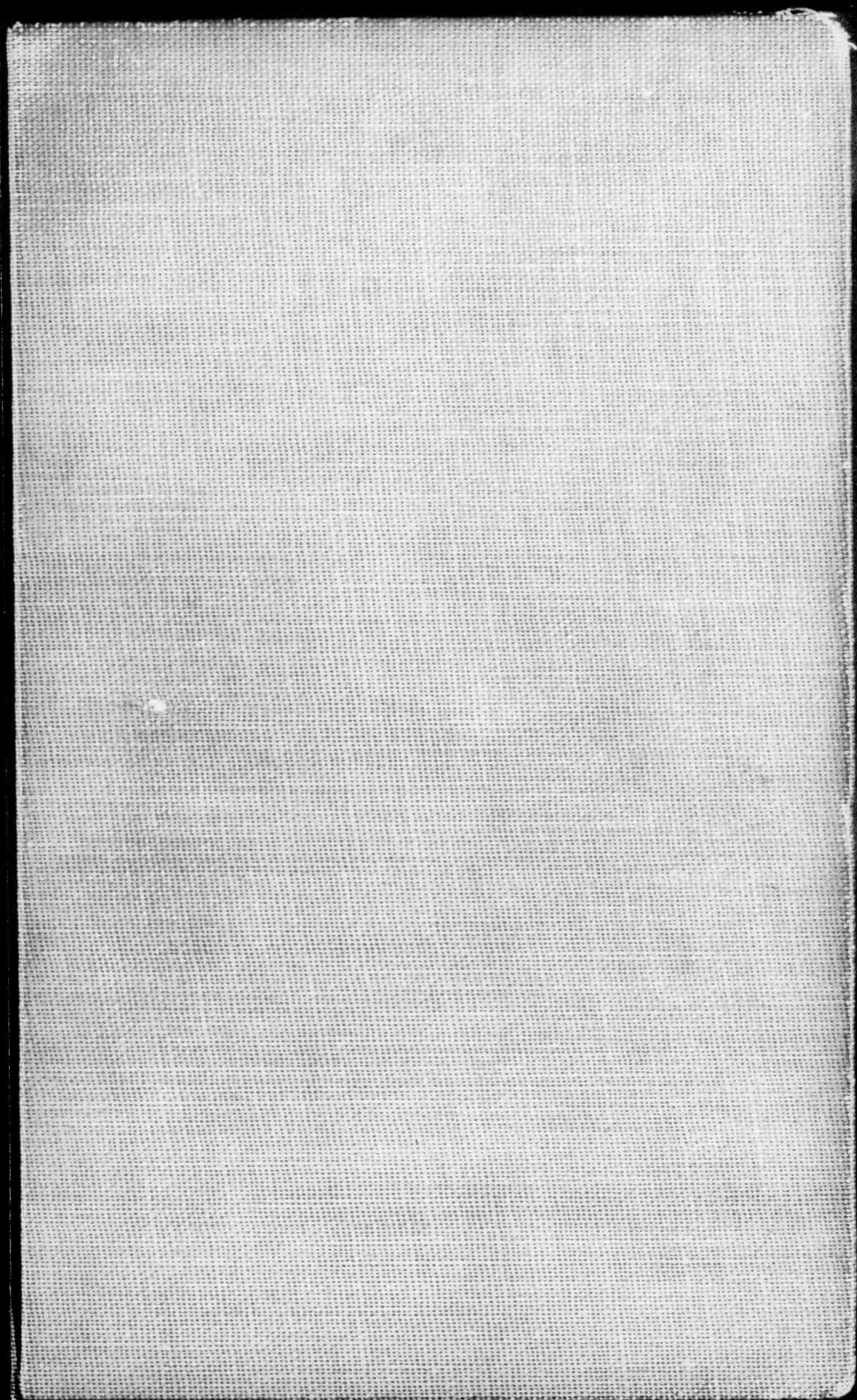
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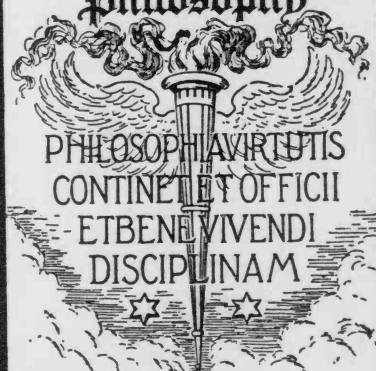
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# LOGIC

## PART II DEMONSTRATIVE INFERENCE: DEDUCTIVE AND INDUCTIVE

BY

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## INTRODUCTION TO PART II

§ I. BEFORE introducing the topics to be examined in Part II, I propose to recapitulate the substance of Part I, and in so doing to bring into connection with one another certain problems which were there treated in different chapters. I hope thus to lay different emphasis upon some of the theories that have been maintained, and to remove any possible misunderstandings where the treatment was unavoidably condensed.

In my analysis of the proposition I have distinguished the natures of substantive and adjective in a form intended to accord in essentials with the doctrine of the large majority of logicians, and as far as my terminology is new its novelty consists in giving wider scope to each of these two fundamental terms. *Prima facie* it might be supposed that the connection of substantive with adjective in the construction of a proposition is tantamount to the metaphysical notions of substance and inherence. But my notion of substantive is intended to include, besides the metaphysical notion of substance—so far as this can be philosophically justified—the notion of occurrences or events to which some philosophers of the present day wish to restrict the realm of reality. Thus by a substantive *proper* I mean an existent; and the category of the existent is divided into the two subcategories: what continues to exist, or the continuant; and what ceases to exist, or the occurrent, every occurrent being referrible to a continuant. To exist is to be

in temporal or spatio-temporal relations to other existents; and these relations between existents are the fundamentally external relations. A substantive proper cannot characterise, but is necessarily characterised; on the other hand, entities belonging to any category whatever (substantive proper, adjective, proposition, etc.) may be characterised by adjectives or relations belonging to a special adjectival sub-category corresponding, in each case, to the category of the object which it characterises. Entities, other than substantives proper, of which appropriate adjectives can be predicated, function as quasi-substantives.

§ 2. The term adjective, in my application, covers a wider range than usual, for it is essential to my system that it should include relations. There are two distinct points of view from which the treatment of a relation as of the same logical nature as an adjective may be defended. In the first place the complete predicate in a relational proposition is, in my view, relatively to the subject of such proposition, equivalent to an adjective in the ordinary sense. For example, in the proposition, 'He is afraid of ghosts,' the relational component is expressed by the phrase 'afraid of'; but the complete predicate 'afraid of ghosts' (which includes this relation) has all the logical properties of an ordinary adjective, so that for logical purposes there is no fundamental distinction between such a relational predicate and an irrational predicate. In the second place, if the relational component in such a proposition is separated, I hold that it can be treated as an adjective predicated of the substantive-couple 'he' and 'ghosts'. In other words, a relation cannot be identified with a *class* of couples, i.e. be

conceived extensionally; but must be understood to *characterise* couples, i.e. be conceived intensionally. It seems to me to raise no controvertible problem thus to include relations under the wide genus adjectives. It is compatible, for example, with almost the whole of Mr Russell's treatment of the proposition in his *Principles of Mathematics*; and, without necessarily entering into the controvertible issues that emerge in such philosophical discussions, I hold that some preliminary account of relations is required even in elementary logic.

§ 3. My distinction between substantive and adjective is roughly equivalent to the more popular philosophical antithesis between particular and universal; the notions, however, do not exactly coincide. Thus I understand the philosophical term particular not to apply to quasi-substantives, but to be restricted to substantives proper, i.e. existents, or even more narrowly to occur-rents. On the other hand, I find a fairly unanimous opinion in favour of calling an adjective predicated of a particular subject, a particular—the name universal being confined to the abstract conception of the adjective. Thus red or redness, abstracted from any specific judgment, is held to be universal; but the redness, manifested in a particular object of perception, to be itself particular. Furthermore, *qua* particular, the adjective is said to be an existent, apparently in the same sense as the object presented to perception is an existent. To me it is difficult to argue this matter because, while acknowledging that an adjective may be called a universal, I regard it not as a mere abstraction, but as a factor in the real; and hence, in holding that the objectively real is properly construed into an adjective



characterising a substantive, the antithesis between the particular and the universal (i.e. in my terminology between the substantive and the adjective) does not involve separation within the real, but solely a separation *for thought*, in the sense that the conception of the substantive apart from the adjective, as well as the conception of the adjective apart from the substantive, equally entail abstraction.

§ 4. Again, taking the whole proposition constituted by the connecting of substantive with adjective, I have maintained that in a virtually similar sense the proposition is to be conceived as abstract. But, whereas the characterising tie may be called constitutive in its function of connecting substantive with adjective to construct the proposition, I have spoken of the assertive tie as epistemic, in the sense that it connects the thinker with the proposition in constituting the unity which may be called an act of judgment or of assertion. When, however, this act of assertion becomes in its turn an object of thought, it is conceived under the category of the existent; for such an act has temporal relations to other existents, and is necessarily referrible to a thinker conceived as a continuant. Though, relatively to the primary proposition, the assertive tie must be conceived as epistemic; yet, relatively to the secondary proposition which predicates of the primary that it has been asserted by A, the assertive tie functions constitutively.

§ 5. In view of a certain logical condition presupposed throughout this Part of my work, I wish to remind the reader of that aspect of my analysis of the proposition, according to which I regard the subject as that which is *given* to be determinately characterised

by thought. Now I hold that for a subject to be characterised by some adjectival determinate, it must first have been presented as characterised by the corresponding adjectival determinable. The fact that what is given is characterised by an adjectival determinable is constitutive; but the fact that it is *presented* as thus characterised is epistemic. Thus, for a surface to be characterised as red or as square, it must first have been constructed in thought as being the kind of thing that has colour or shape; for an experience to be characterised as pleasant or unpleasant, it must first have been constructed in thought as the kind of thing that has hedonic tone. Actually what is given, is to be determined with respect to a conjunction of several specific aspects or determinables; and these determine the category to which 'the given' belongs. For example, on the dualistic view of reality, the physical has to be determined under spatio-temporal determinables, and the psychical under the determinable consciousness or experience. If the *same* being can be characterised as two-legged and as rational, he must be put into the category of the physico-psychical.

§ 6. The passage from topics treated in Part I to those in Part II, is equivalent to the step from implication to inference. The term inference, as introduced in Part I, did not require technical definition or analysis, as it was sufficiently well understood without explanation. It was, however, necessary in Chapter III to indicate in outline one technical difficulty connected with the paradox of implication; and there I first hinted, what will be comprehensively discussed in the first chapter of this Part, that implication is best conceived

as potential inference. While for elementary purposes implication and inference may be regarded as practically equivalent, it was pointed out in Chapter III that there is nevertheless one type of limiting condition upon which depends the possibility of using the relation of implication for the purposes of inference. Thus reference to the specific problem of the paradox of implication was unavoidable in Part I, inasmuch as a comprehensive account of symbolic and mechanical processes necessarily included reference to all possible limiting cases; but, apart from such a purely abstract treatment, no special logical importance was attached to the paradox. The limiting case referred to was that of the permissible employment of the compound proposition 'If  $p$  then  $q$ ,' in the unusual circumstance where knowledge of the truth or the falsity of  $p$  or of  $q$  was already present when the compound proposition was asserted. This limiting case will not recur in the more important developments of inference that will be treated in the present part of my logic.

It might have conduced to greater clearness if, in Chapters III and IV, I had distinguished—when using the phrase *implicative proposition*—between the primary and secondary interpretations of this form of proposition. Thus, when the compound proposition 'If  $p$  then  $q$ ' is rendered, as Mr Russell proposes, in the form 'Either not- $p$  or  $q$ ,' the compound is being treated as a *primary* proposition of the same type as its components  $p$  and  $q$ . When on the other hand we substitute for 'If  $p$  then  $q$ ' the phrase ' $p$  implies  $q$ ,' or preferably ' $p$  would imply  $q$ ,' the proposition is no longer primary, inasmuch as it predicates about the proposition  $q$  the adjective 'implied by  $p$ ' which renders the compound a secondary

proposition, in the sense explained in Chapter IV<sup>1</sup>. Now whichever of these two interpretations is adopted, the inference which is legitimate under certain limiting conditions is the same. Thus given the compound 'Either not- $p$  or  $q$ ' conjoined with the assertion of ' $p$ ,' we could infer ' $q$ '; just as given ' $p$  implies  $q$ ' conjoined with the assertion of ' $p$ ,' we infer ' $q$ .' It is for this reason that the two interpretations have become merged into one in the ordinary symbolic treatment of compound propositions; and in normal cases no distinction is made in regard to the possibility of using the primary or secondary interpretation for purposes of inference. The normal case, however, presupposes that  $p$  and  $q$  are entertained hypothetically; when this does not obtain, the danger of *petitio principii* enters. The problem in Part I was only a very special and technical case in which this fallacy has to be guarded against; in Part II, it will be dealt with in its more concrete and philosophically important applications.

§ 7. The mention of this fallacy immediately suggests Mill's treatment of the functions and value of the syllogism; but, before discussing his views, I propose to consider what his main purpose was in tackling the charge of *petitio principii* that had been brought against the whole of formal argument, including in particular the syllogism. In the first section of his chapter, Mill refers to two opposed classes of philosophers—the one of whom regarded syllogism as the universal type of all logical reasoning, the other of whom regarded syllogism

<sup>1</sup> The interpretation of the implicative form ' $p$  implies  $q$ ' as secondary is developed in Chapter III, § 9, where the modal adjectives *necessary*, *possible*, *impossible*, are introduced.

as useless on the ground that all such forms of inference involve *petitio principii*. He then proceeds: 'I believe both these opinions to be fundamentally erroneous,' and this would seem to imply that he proposed to relieve the syllogism from the charge. I believe, however, that all logicians who have referred to Mill's theory—a group which includes almost everyone who has written on the subject since his time—have assumed that the purport of the chapter was to *maintain* the charge of *petitio principii*, an interpretation which his opening reference to previous logicians would certainly not seem to bear. His subsequent discussion of the subject is, verbally at least, undoubtedly confusing, if not self-contradictory; but my personal attitude is that, whatever may have been Mill's general purpose, it is from his own exposition that I, in common with almost all his contemporaries, have been led to discover the principle according to which the syllogism can be relieved from the incubus to which it had been subject since the time of Aristotle. In my view, therefore, Mill's account of the philosophical character of the syllogism is incontrovertible; I would only ask readers to disregard from the outset any passage in his chapter in which he appears to be contending for the annihilation of the syllogism as expressive of any actual mode of inference.

Briefly his position may be thus epitomised. Taking a typical syllogism with the familiar major 'All men are mortal,' he substituted for 'Socrates' or 'Plato' the minor term 'the Duke of Wellington' who was then living. He then maintained that, going behind the syllogism, certain instantial evidence is required for establishing the major; and furthermore that the validity

of the conclusion that the Duke of Wellington would die depends ultimately on this instantial evidence. The interpolation of the universal major 'All men will die' has undoubted value, to which Mill on the whole did justice; but he pointed out that the formulation of this universal adds nothing to the positive or factual data upon which the conclusion depends. It follows from his exposition that a syllogism whose major is admittedly established by induction from instances can be relieved from the reproach of begging the question or circularity if, and only if, the minor term is not included in the ultimate evidential data. The Duke of Wellington being still living could not have formed part of the evidence upon which the universal major depended. It was therefore part of Mill's logical standpoint to maintain that there were principles of induction by which, from a limited number of instances, a universal going beyond these could be logically justified. This contention may be said to confer constitutive validity upon the inductive process. It is directly associated with the further consideration that an instance, not previously examined, may be adduced to serve as minor premiss for a syllogism, and that such an instance will always preclude circularity in the formal process. Now the charge of circularity or *petitio principii* is epistemic; and the whole of Mill's argument may therefore be summed up in the statement that the epistemic validity of syllogism and the constitutive validity of induction, both of which had been disputed by earlier logicians, stand or fall together.

In order to prevent misapprehension in regard to Mill's view of the syllogism, it must be pointed out that he virtually limited the topic of his chapter to cases in

which the major premiss would be admitted by all logicians to have been established by means of induction in the ordinary sense, i.e. by the simple enumeration of instances; although many of them would have contended that such instancial evidence was not by itself sufficient. Thus all those cases in which the major was otherwise established, such as those based on authority, intuition or demonstration, do not fall within the scope of Mill's solution. Unfortunately all the commentators of Mill have confused his view that universals cannot be intuitively but only empirically established, with his specific contention in Chapter IV. I admit that he himself is largely responsible for this confusion, and therefore, while supporting his view on the functions of the syllogism, I must deliberately express my opposition to his doctrine that universals can only ultimately be established empirically, and limit my defence to his analysis of those syllogisms in which it is acknowledged that the major is thus established. Even here his doctrine that all inference is from particulars to particulars is open to fundamental criticism; and, in my treatment of the principles of inductive inference which will be developed in Part III, I shall substitute an analysis which will take account of such objections as have been rightly urged against Mill's exposition.

[NOTE. There are two cases in which the technical terminology employed in Part II differs from that in Part I. (1) The phrase *primitive proposition*, in Part I, is to be understood psychologically; in Part II, logically as equivalent to *axiom*. (2) *Counter-implicative*, in Part I, applies to the form of a compound proposition; in Part II, to a principle of inference.]

## CHAPTER I

### INFERENCE IN GENERAL

§ 1. INFERENCE is a mental process which, as such, has to be contrasted with implication. The connection between the mental act of inference and the relation of implication is analogous to that between assertion and the proposition. Just as a proposition is what is potentially assertible, so the relation of implication between two propositions is an essential condition for the possibility of inferring one from the other; and, as it is impossible to define a proposition ultimately except in terms of the notion of asserting, so the relation of implication can only be defined in terms of inference. This consideration explains the importance which I attach to the recognition of the mental attitude involved in inference and assertion; after which the strictly logical question as to the distinction between valid and invalid inference can be discussed. To distinguish the formula of implication from that of inference, the former may be symbolised 'If  $p$  then  $q$ ,' and the latter ' $p$  therefore  $q$ ,' where the symbol  $q$  stands for the conclusion and  $p$  for the premiss or conjunction of premisses.

The proposition or propositions from which an inference is made being called premisses, and the proposition inferred being called the conclusion, it is commonly supposed that the premisses are the propositions first presented in thought, and that the transition from these to the thought of the conclusion is the

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The proposition or propositions from which an inference is made being called premisses, and the proposition inferred being called the conclusion, it is commonly supposed that the premisses are the propositions first presented in thought, and that the transition from these to the thought of the conclusion is the



last step in the process. But in fact the reverse is usually the case; that is to say, we first entertain in thought the proposition that is technically called the conclusion, and then proceed to seek for other propositions which would justify us in asserting it. The conclusion may, on the one hand, first present itself to us as potentially assertible, in which case the mental process of inference consists in transforming what was *potentially* assertible into a proposition *actually* asserted. On the other hand, we may have already satisfied ourselves that the conclusion can be validly asserted apart from the particular inferential process, in which case we may yet seek for other propositions which, functioning as premisses, would give an independent or additional justification for our original assertion. In every case, the process of inference involves three distinct assertions: first the assertion of '*p*,' next the assertion of '*q*,' and thirdly the assertion that '*p* would imply *q*.' It must be noted that '*p* would imply *q*,' which is the proper equivalent of 'if *p* then *q*,' is the more correct expression for the relation of implication, and not '*p* implies *q*'—which rather expresses the completed inference. This shows that inference cannot be defined in terms of implication, but that implication must be defined in terms of inference, namely as equivalent to potential inference. Thus, in inferring, we are not merely passing from the assertion of the premiss to the assertion of the conclusion, but we are also implicitly asserting that the assertion of the premiss is *used* to justify the assertion of the conclusion.

§ 2. Some difficult problems, which are of special importance in psychology, arise in determining quite

precisely the range of those mental processes which may be called *inference*: in particular, how far assertion or inference is involved in the processes of association and of perception. These difficulties have been aggravated rather than removed by the quite false antithesis which some logicians have drawn between logical and psychological inference. Every inference is a mental process, and therefore a proper topic for psychological analysis; on the other hand, to infer is to think, and to think is virtually to adopt a logical attitude; for everyone who infers, who asserts, who thinks, *intends* to assert truly and to infer validly, and this is what constitutes assertion or inference into a logical process. It is the concern of the science of logic, as contrasted with psychology, to criticise such assertions and inferences from the point of view of their validity or invalidity.

Let us then consider certain mental processes—in particular processes of association—which have the semblance of inference. In the first place, there are many unmistakeable cases of association in which no inference whatever is even *apparently* involved. Any familiar illustration, either of contiguity or of similarity, will prove that association in itself does not entail inference. If a cloudy sky raises memory-images of a storm, or leads to the mental rehearsal of a poem, or suggests the appearance of a slate roof, in none of these revivals by association is there involved anything in the remotest degree resembling inference. The case of contiguity is that which is most commonly supposed to involve some sort of inference; but in this supposal there is a confusion between recollection and expectation. Our recollection of storms that we have experienced in



the past is obviously distinct from our expectation that a storm is coming on in the immediate future. It is to this latter process of expectation, and not to the former process of recollection, that the term inference is more or less properly applied; but even here we must make a careful psychological distinction. We may expect a storm when we notice the darkness of the sky, without at all having actually recalled past experiences of storms; in this case no inference is involved, since there has been only one assertion, namely, what would constitute the conclusion without any other assertion that would constitute a premiss. In order to speak properly of inference in such cases, the minimum required is the assertion that the sky is cloudy and that *therefore* there will be a storm. Here we have two explicit assertions, together with the inference involved in the word 'therefore.' It is of course a subtle question for introspection as to whether this threefold assertion really takes place. This difficulty does not at all affect our definition of inference; it would only affect the question whether in any given case inference had actually occurred. It has been suggested that, where there has been nothing that logic could recognise as an inference, there has yet been inference in a psychological sense; but this contention is absurd, since it is entirely upon psychological grounds that we have denied the existence of inference in such cases.

Let us consider further the logical aspects of a genuine inference, following upon such a process of association as we have illustrated. The scientist may hold that the appearance of the sky is not such as to warrant the expectation of an on-coming storm. He

may, therefore, criticise the inference as invalid. Thus, assuming the actuality of the inference from the psychological point of view, it may yet be criticised as invalid from the logical point of view. So far we have taken the simplest case, where the single premiss 'The sky is cloudy' is asserted. But, when an additional premiss such as 'In the past cloudy skies have been followed by storm' is asserted, then the inference is further rationalised, since the two premisses taken together constitute a more complete ground for the conclusion than the single premiss. This additional premiss is technically known as a *particular* proposition. If the thinker is pressed to find still stronger logical warrant for his conclusion, he may assert that in *all* his experiences cloudy skies have been followed by storm (a limited universal). The final stage of rationalisation is reached when the universal limited to all remembered cases is used as the ground for asserting the *unlimited* universal for all cases. But even now the critic may press for further justification. To pursue this topic would obviously require a complete treatment of induction, syllogism, etc., from the logical point of view. Enough has been said to show that, however inadequate may be the grounds offered in justification of a conclusion, this has no bearing upon the nature or upon the fact of inference as such, but only upon the criticism of it as valid or invalid.

As in association, so also in perception, a psychological problem presents itself. There appear to be at least three questions in dispute regarding the nature of perception, which have close connection with logical analysis: First, how much is contained in the percept

besides the immediate sense experience? Secondly, does perception involve assertion? Thirdly, does it involve inference? To illustrate the nature of the first problem, let us consider what is meant by the visual perception of a match-box. This is generally supposed to include the representation of its tactual qualities; in which case, the content of the percept includes qualities other than those sensorially experienced. On the other hand, supposing that an object touched in the dark is recognised as a match-box, through the special character of the tactual sensations, would the *representation* of such visual qualities as distinguish a match-box from other objects be included in the tactual perception of it as a match-box? The same problem arises when we recognise a rumbling noise as indicating a cart in the road: i.e. should we say, in this case, that the auditory percept of the cart includes visual or other distinguishing characteristics of the cart not sensorially experienced? In my view it is inconsistent to include in the content of the visual percept tactual qualities not sensorially experienced, unless we also include in the content of a tactual or auditory percept visual or similar qualities not sensorially experienced<sup>1</sup>.

This leads up to our second question, namely whether in such perceptions there is an assertion (*a*) predicating of the experienced sensation certain specific qualities; or an assertion (*b*) of having experienced in the past similar sensations simultaneously with the perception of

<sup>1</sup> In speaking here of the *mental representation* of qualities not sensorially experienced, I am putting entirely aside the very important psychological question as to whether such mental representations are in the form of 'sense-imagery' or of 'ideas.'

a certain object. Employing our previous illustration, we may first question whether the assertion 'There is a cart in the road' following upon a particular auditory sensation, involves (*a*) the explicit characterisation of that sensation. Now if the specific character of the noise as a sensation merely *caused* a visual image which in its turn caused the assertion 'There is a cart in the road,' then in the absence of assertion (*a*) there is no explicit inference. In order to become inference, the character operating (through association) as *cause* would have to be predicated (in a connective judgment) as *ground*. On the other hand, any experience that could be described as hearing a noise of a certain more or less determinate character would involve, in my opinion, besides assimilation, a judgment or assertion (*a*) expressible in some such words as 'There is a rumbling noise.' The further assertion that there is a cart in the road is accounted for (through association) by previous experiences of hearing such a noise simultaneously with seeing a cart. Assuming that association operates by arousing memory-images of these previous experiences, it is only when by their vividness or obtrusiveness these memory-images give rise to a *memory-judgment*, that the assertion (*b*) occurs. We are now in a position to answer the third question as to the nature of perception; for, if either the assertion of (*a*) alone or of (*b*) with (*a*) occurs along with the assertion that there is a cart in the road, then inference is involved; otherwise it is not.

§ 3. Passing from the psychological to the strictly logical problem, we have to consider in further detail the conditions for the validity of an inference symbolised as '*p* ∴ *q*.' These conditions are twofold, and may be

conveniently distinguished in accordance with my terminology as constitutive and epistemic. They may be briefly formulated as follows:

*Conditions for Validity of the Inference 'p ∴ q'*

*Constitutive Conditions:* (i) the proposition 'p' and (ii) the proposition 'p would imply q' must both be true.

*Epistemic Conditions:* (i) the asserting of 'p' and (ii) the asserting of 'p would imply q' must both be permissible without reference to the asserting of q.

It will be noted that the constitutive condition exhibits the dependence of inferential validity upon a certain relation between the *contents* of premiss and of conclusion; the epistemic condition, upon a certain relation between the *asserting* of the premiss and the *asserting* of the conclusion. Taking the constitutive condition first, we observe that the distinction between inference and implication is sometimes expressed by calling implication 'hypothetical inference'—the meaning of which is that, in the act of inference, the premiss must be categorically asserted; while, in the relation of implication, this premiss is put forward merely hypothetically. This was anticipated above by rendering the relation of implication in the subjunctive mood (p would imply q) and the relation of inference in the indicative mood (p implies q).

Further to bring out the connection between the epistemic and the constitutive conditions, it must be pointed out that an odd confusion attaches to the use of the word 'imply' in these problems. The almost universal application of the relation of implication in logic is as a relation between two propositions; but, in familiar language, the term 'imply' is used as a relation

between two assertions. Consider for instance (a) 'B's asserting that there will be a thunderstorm would imply his having noticed the closeness of the atmosphere,' and (b) 'the closeness of the atmosphere would imply that there will be a thunderstorm.' The first of these relates two mental acts of the general nature of assertion, and is an instance of 'the asserting of q would imply having asserted p'; the second is a relation between two propositions, and is an instance of 'the proposition p would imply the proposition q.' Comparing (a) with (b) we find that implicans and implicate have changed places. Indeed the sole reason why the asserting of the thunderstorm was supposed to imply having asserted the closeness of the atmosphere was that, in the speaker's judgment, the closeness of the atmosphere would imply that there will be a thunderstorm.

Recognising, then, this double and sometimes ambiguous use of the word 'imply,' we may restate the first of the two epistemic conditions and the second of the two constitutive conditions for the validity of the inference 'p ∴ q' as follows:

*Epistemic condition* (i): the asserting of the proposition 'p' should *not* have implied the asserting of the proposition 'q.'

*Constitutive condition* (ii): the proposition 'p' should imply the proposition 'q.'

The former is merely a condensed equivalent of our original formulation, viz. that 'the asserting of the proposition 'p' must be permissible without reference to the asserting of the proposition 'q.'

Now the fact that there is this double use of the term 'imply' accounts for the paradox long felt as

regards the nature of inference: for it is urged that, in order that an inference may be formally valid, it is required that the conclusion should be *contained* in the premiss or premisses; while, on the other hand, if there is any genuine advance in thought, the conclusion must *not* be contained in the premiss. This word 'contained' is doubly ambiguous: for, in order to secure formal validity, the premisses regarded as propositions *must imply* the conclusion regarded as a proposition; but, in order that there shall be some real advance and not a mere *petitio principii*, it is required that the asserting of the premisses *should not have* implied the previous asserting of the conclusion. These two horns of the dilemma are exactly expressed in the constitutive and epistemic conditions above formulated.

§ 4. We shall now explain how the constitutive conditions for the validity of inference, which have been expressed in their most general form, are realised in familiar cases. The general constitutive condition '*p* would imply *q*' is *formally* satisfied when some specific logical relation holds of *p* to *q*; and it is upon such a relation that the formal truth of the assertion that '*p* would imply *q*' is based. There are two fundamental relations which will render the inference from *p* to *q*, not only valid, but *formally* valid; and these relations will be expressed in formulae exhibiting what will be called the *Applicative* and the *Implicative* Principles of Inference. The former may be said to formulate what is involved in the intelligent use of the word 'every'; the latter what is involved in the intelligent use of the word 'if.'

In formulating the Applicative principle, we take *p*

to stand for a proposition universal in form, and *q* for a singular proposition which predicates of some single case what is predicated universally in *p*. The Applicative principle will then be formulated as follows:

From a predication about 'every' we may formally infer the same predication about 'any given.'

In formulating the Implicative principle, we take *p* to stand for a compound proposition of the form '*x* and "*x* implies *y*"' and *q* to stand for the simple proposition '*y*.' The Implicative principle will then be formulated as follows:

From the compound proposition '*x* and "*x* implies *y*"' we may formally infer '*y*.'

§ 5. We find two different forms of proposition, one or other of which is used as a premiss in every formal inference; the distinction between which is fundamental, but has been a matter of much controversy among logicians. In familiar logic the two kinds of proposition to which I shall refer are known respectively as universal and hypothetical. As an example of the former, take 'Every proposition can be subjected to logical criticism'; from this universal proposition we may directly infer 'That "*matter exists*" can be subjected to logical criticism.' This inference illustrates what I have called the Applicative Principle, and its premiss will be called an Applicative universal. Take next the example 'If this can swim it breathes,' and 'it can swim'; from this conjunction of propositions we infer that 'it breathes'; here, the hypothetical premiss being in our terminology called implicative, the inference in question illustrates the use of the Implica-

tive Principle. It is the combination of these two principles that marks the advance made in passing from the most elementary forms of inference to the syllogism. For example: From 'Everything breathes if able to swim' we can infer 'This breathes if able to swim,' where the applicative principle only is employed. Conjoining the conclusion thus obtained with the further premiss 'This can swim,' we can infer 'this breathes,' where the implicative principle only is employed. In this analysis of the syllogism which involves the interpolation of an additional proposition, we have shown how the two principles of inference are successively employed. The ordinary formulation of the syllogism would read as follows: 'Everything that can swim breathes; this can swim; therefore this breathes.' In place of the usual expression of the major premiss, I have substituted 'Everything breathes if able to swim,' in order to show how the major premiss prepares the way for the inferential employment successively of the applicative and of the implicative principles.

§ 6. Now the two propositions 'Every proposition can be subjected to logical criticism' and 'everything that is able to swim breathes' must be carefully contrasted. Both of them are universal in form; but in the latter the subject term contains an explicit characterising adjective, viz. *able to swim*. The presence of a characterising adjective in the subject anticipates the occasion on which the question would arise whether this adjective is to be predicated of a given object. In the syllogism, completed as in the preceding section, the universal major premiss is combined with an affirmative minor premiss, where the adjective entertained cate-

gorically as *predicate* of the minor is the same as that which was entertained hypothetically as *subject* of the major. This double functioning of an adjective is the one fundamental characteristic of all syllogism; where it will be found that one (or, in the fourth figure, every) term occurs once in the subject of a proposition, where it is entertained hypothetically, and again in the predicate of another proposition where it is entertained categorically.

The essential distinction between the two contrasted universals (applicational and implicational) lies in the fact that an inference can be drawn from the former on the applicative principle alone, which dispenses with the minor premiss. We have to note the nature of the substantive that occurs in the *applicational* universal as distinguished from that which occurs in the *implicational* universal. The example already given contained 'proposition' as the subject term, and a few other examples are necessary to establish the distinction in question. 'Every individual is self-identical,' therefore 'the author of the *Republic* is self-identical'; 'Every conjunction of predications is commutative,' therefore 'the conjunction *lightning before* and *thunder after* is commutative'; 'Every adjective is a relatively determinate specification of a relatively indeterminate adjective,' therefore '*red* is a relatively determinate specification of a relatively indeterminate adjective.' These illustrations could be endlessly multiplied, in which we directly apply a universal proposition to a certain given instance. In such cases the implicative as well as the applicative principle would have been involved if it had been necessary or possible to interpolate, as an additional



datum, a categorical proposition requiring certification, to serve as minor premiss. Let us turn to our original illustration and examine what would have been involved if we had treated the inference as a syllogism; it would have read as follows: 'Every proposition can be subjected to logical criticism'; 'That *matter exists* is a proposition'; therefore 'That *matter exists* can be subjected to logical criticism.' In this form, the substantive word *proposition* occurs as subject in the universal premiss, and as predicate in the singular premiss. What I have to maintain is that this introduction of a minor premiss is superfluous and even misleading. It should be observed that, in all the illustrations given above of the purely applicative principle, the subject-term in the universal premiss denotes a general category. It follows from this that the proposed statement 'That *matter exists* is a proposition' is redundant as a premiss; for it is impossible for us to understand the meaning of the phrase 'matter exists' except so far as we understand it to denote a proposition. In the same way, it would be impossible to understand the word 'red' without understanding it to denote an adjective; and so in all other cases of the pure employment of the applicative principle. In all these cases, the minor premiss which might be constructed is not a genuine proposition—the truth of which could come up for consideration—because the understanding of the subject-term of the minor demands a reference of it to the general category there predicated of it. This proposed minor premiss, therefore, is a peculiar kind of proposition which is not exactly what Mill calls 'verbal,' but rather what Kant meant by 'analytic,' and which I propose to call '*struc-*

*tural*.' All structural statements contain as their predicate some wide logical category, and their fundamental characteristic is that it is impossible to realise the meaning of the subject-term without implicitly conceiving it under that category. The structural proposition can hardly be called verbal, because it does not depend upon any arbitrary assignment of meaning to a word;—this point being best illustrated by giving examples. For instance, taking as subject-term 'the author of the *Republic*,' then 'The author of the *Republic* wrote something,' would be verbal, while 'The author of the *Republic* is an individual,' would be structural. In reality the subject of a verbal proposition, and the subject of a structural proposition are not the same; the one has for its subject the *phrase* 'the author of the *Republic*,' and the other the *object* denoted by the phrase. This is the true and final principle for distinguishing a structural (as well as a genuinely real or synthetic statement) from a verbal statement.

§ 7. Since a category is expressed always by a general substantive name, the important question arises as to whether or how the name of a category such as 'existent' or 'proposition' is to be defined. Now the ordinary general substantive name is defined in terms of *determinate* adjectives which constitute its connotation; but, so far as a category can be defined, it must be in terms of adjectival *determinables*; e.g. an existent is what occupies some region of space or period of time: the determinates corresponding to which would be, occupying some specific region of space or period of time. Similarly, the category 'proposition' could be defined by the adjectival determinable 'that to which



some assertive attitude can be adopted,' under which the relative determinates would be *affirmed, denied, doubted*, etc. We may indicate the nature of a given category by assigning the determinables involved in its construction. Using capital letters for determinables and corresponding small letters for their determinates (distinguished amongst themselves by dashes), the major premiss of the syllogism would assume the following form: Every  $MP$  is  $p$  if  $m$ ; where the determinables  $M$  and  $P$  serve to define the category so far as required for the syllogism in question. Here we substitute for the vague word 'thing' previously employed, the symbol  $MP$  to indicate the category of reference; namely, that comprising substantives of which some determinate character under the determinables  $M$  and  $P$  can be predicated. The statement that the given thing is  $MP$  is redundant where  $M$  and  $P$  are determinables to which the given thing belongs; for the thing could not be given either immediately or in an act of construction except so far as it was given under the category defined by these determinables. Hence any genuine act of characterisation of the thing so given would consist in giving to these mere determinables a comparatively determinate value. For example, it being assumed that the given thing is  $MP$ , we may characterise it in such determinate forms as ' $m$  and  $p$ ,' ' $m$  or  $p$ ,' ' $p$  if  $m$ ,' 'not both  $p$  and  $m$ ,' where the predication of the relative determinates  $m$  and  $p$  would presuppose that the object had been constructed under  $MP$ . In defining the function of a proposition to be to characterise relatively determinately what is *given* to be characterised, we now see that what is '*given*' is not given in a merely abstract

sense, but—in being given—the determinables which have to be determined are already presupposed.

§ 8. We may now show more clearly why the force of the term 'every' is distinct from that of the term 'if'; and how, in the syllogism, the two corresponding principles of inference are both involved. The major premiss having been formulated in terms of the determinables  $M$  and  $P$ , the whole argument will assume the following form:

(a) Every  $MP$  is  $p$  if  $m$ ,

from which we infer, by the applicative principle alone:

(b) The given  $MP$  is  $p$  if  $m$ .

Next we introduce the minor, viz.

(c) The given  $MP$  is  $m$ ,

and finally infer, by the implicative principle alone:

(d) The given  $MP$  is  $p$ .

Now if we held that the inference from (a) to (b) required the implicative principle as well as the applicative, so that a minor premiss 'The given thing is  $MP$ ' must be interpolated, the syllogism would assume the following more complicated form:

(a) Everything is  $p$  if  $m$  if  $MP$  (the reformulated major).

∴ (b) The given thing is  $p$  if  $m$  if  $MP$  (by the applicative principle alone).

Next we introduce as minor

(c) The given thing is  $MP$ .

∴ (d) The given thing is  $p$  if  $m$  (by the implicative principle alone);

finally, introducing the original minor, viz.

(e) The given thing is  $m$ .

∴ (f) The given thing is  $p$  (by the implicative principle alone).

Now this lengthened analysis of the syllogism, while involving the implicative principle twice, involves as well as the applicative principle the introduction of a new minor, viz. that the given thing is *MP*, which hints at the doubt whether what is given is given as *MP*. But if this were a reasonable matter of doubt requiring explicit affirmation, on the same principle we might doubt whether what is given is a 'thing,' in some more generic sense of the word 'thing.' If this doubt be admitted, the syllogism is resolved into *three* uses of the implicative principle, with *two* extra minor premisses. Such a resolution would in fact lead by an infinite regress to an infinite number of employments of the implicative principle. To avoid the infinite regress we must establish some principle for determining the point at which an additional minor is not required. The view then that I hold is not merely that what is given is a 'thing' in the widest sense of the term thing, but that what is given is always given as demanding to be characterised in certain definite respects—e.g. colour, size, weight; or cognition, feeling, conation—and that therefore such a proposition as 'The given thing is *MP*' is presupposed in its being given, i.e. in being given, it is given as requiring determination with respect to these definite determinables *M* and *P*. The above formulation, therefore, in which the syllogism is resolved into a process involving the applicative and the implicative principles each *only once*, is logically justified; for it brings out the distinction between the function of the term *every* as leading to the employment of the applicative principle alone, and the function of *if* as leading to the employment of the implicative principle

alone; and furthermore it distinguishes between the process in inference which requires the applicative principle alone from that which requires the implicative as well as the applicative principle.

The distinction between the cases in which the implicative principle can or cannot be dispensed with depends, so far, upon whether the subject-term of the universal stands for a logical category or not. But we may go further and say that, even if the subject of the universal is not a logical category, provided that it is definable by certain determinables, and that the subject of the conclusion is only apprehensible under those determinables, then again the use of the implicative principle may be dispensed with. For example: 'All material bodies attract; therefore, the earth attracts.' Here the term 'material body' is of the nature of a category in that it can only be defined under such determinables as 'continuing to exist' and 'occupying some region of space'; furthermore the earth is constructively given under these determinables: hence a proposed minor premiss to the effect that the earth is a material body is superfluous, and the above inference involves only the applicative principle. Again 'All volitional acts are causally determined; therefore, Socrates' drinking of hemlock was causally determined.' Here the subject of the conclusion is constructively given under the determinables involved in the definition of volitional act, which again justifies the use of the applicative principle alone. As a third example: 'Every denumerable aggregate is less than some other aggregate: therefore, an aggregate whose number is  $\aleph_0$  is numerically less than some other aggregate.' Here the construction of the

notion of a class whose number is  $\aleph_0$ , involves its being denumerable, so that the given inference again requires only the immediate employment of the applicative principle.

§ 9. Incidentally the above analysis of the major premiss—Every *MP* is *p* if *m*—(or still more simply, 'Every *M* is *m*,' which may sometimes be true; or again, of the minor premiss—'The given *MP* is *m*' or 'The given *MP* is *p*')—accounts for the insistence by certain philosophers, notably Mr Bradley, that every proposition employs the relation of identity; i.e. that the adjective involved in the subject is the same as that involved in the predicate. This philosophical suggestion is, I hold, true, in the sense that the adjectival *determinable* in the subject is the same as that in the predicate; but the latter is a further determination of the former. Now, in this admission that the relation of identity of subject to predicate is involved in the general categorical proposition, I am not in any way withdrawing what was maintained as regards *identity* in my analysis of the proposition. For the identity which I denied was (as it has been expressed) identity in denotation with diversity of connotation, i.e. substantival identity with adjectival diversity. The identity I have accepted above is identity of an *adjectival* factor in the subject with an *adjectival* factor in the predicate. Moreover I should still deny that the proposition *asserts* this identity, and maintain that it simply *presupposes* it, in just the same way as a proposition presupposes the understanding of the meaning of the terms involved without asserting such meaning.

§ 10. We have discussed the case in which a *minor*

premiss may be dispensed with, namely that in which a certain mode of using the applicative principle is sufficient without the employment of the *implicative*. We will now turn to a complementary discussion of the case in which there is unnecessary employment of the *applicative* principle, entailed by the insertion of what may be called a redundant *major* premiss. It will be convenient to call the redundant minor premiss a sub-minor, and the redundant major premiss—to which we shall now turn—a super-major. In this connection I shall introduce the notion of a formal principle of inference, which will apply, not only to inferences that are *strictly* formal, but also to inferences of an inductive nature, for which the principle has not at present been finally formulated and must therefore be here expressed without qualifying detail. The discussion will deal with cases in which the relation of premiss or premisses to conclusion is such that the inference exhibits a formal principle.

We shall illustrate the point first by taking the principle of syllogism, and next, the ultimate (but as yet unformulated) principle of induction. As regards the syllogism, taking *p* and *q* to represent the premisses and *r* the conclusion, we may say that the syllogistic principle asserts that provided a certain relation holds between the three propositions *p*, *q*, and *r*, inference from the premisses *p* and *q* *alone* will formally justify the conclusion *r*. Now it might be supposed that this syllogistic principle constitutes in a sense an additional premiss which, when joined with *p* and *q*, will yield a more complete analysis of the syllogistic procedure. But on consideration it will be seen that there is a sort

of contradiction in taking this view: for the syllogistic principle asserts that the premisses  $p$  and  $q$  are *alone* sufficient for the formal validity of the inference, so that, if the principle is inserted as an additional premiss co-ordinate with  $p$  and  $q$ , the principle itself is virtually contradicted. In illustration we will formulate the syllogistic principle:

‘What can be predicated of every member of a class, to which a given object is known to belong, can be predicated of that object.’

Now, taking a specific syllogism:

‘Every labiate is square-stalked,  
The dead-nettle is a labiate,  
∴ The dead-nettle is square-stalked,’

if we inserted the above-formulated principle as a premiss, co-ordinate with the two given premisses, with a view to strengthening the validity of the conclusion, this would entail a contradiction; because the principle claims that the two premisses are *alone* sufficient to justify the conclusion ‘The dead-nettle is square-stalked.’

Now the same holds, *mutatis mutandis*, of any proposed ultimate inductive principle. Here the premisses are counted—not as *two*—but as many, and summed up in the single proposition ‘All examined instances characterised by a certain adjective are characterised by a certain other adjective’; and the conclusion asserted (with a higher or lower degree of probability) predicates of *all* what was predicated in the premiss of *all examined*. Now, in accordance with the inductive principle, the summary premiss is *sufficient* for asserting the unlimited universal (with a higher or lower degree of probability). To insert this principle, as an additional

premiss co-ordinate with the summary premiss, would, therefore, virtually involve a contradiction. In illustration, we will roughly formulate the inductive principle:

‘What can be predicated of all examined members of a class can be predicated, with a higher or lower degree of probability, of all members of the class.’

Now, taking a specific inductive inference:

‘All examined swans are white. ∴ With a higher or lower degree of probability, all swans are white,’  
if we inserted the above-formulated inductive principle as a premiss, co-ordinate with the summary premiss ‘All examined swans are white,’ with a view to strengthening the validity of the conclusion, this would entail a contradiction; because the principle claims that this summary premiss is *alone* sufficient to justify the conclusion that ‘With a higher or lower degree of probability, all swans are white.’

We may shortly express the distinction between a principle and a premiss by saying that we draw the conclusion *from* the premisses *in accordance with* (or through) the principle. In other words, we immediately see that the relation amongst the premisses and conclusion is a specific case of the relation expressed in the principle, and hence the function of the principle is to stand as a universal to the specific inference as an instance of that universal: where the latter may be said to be inferred from the former (if there is any genuine inference) in accordance with the Supreme Applicative principle. For example: from  $x=y$  and  $y=z$ , we may infer  $x=z$ . This form of inference is expressed, in general terms, in the Principle: ‘Things that are equal to the same thing are equal to one another.’ Now, here,

the two premisses— $x=y$  and  $y=z$ —are *alone* sufficient for the conclusion  $x=z$ ; the conclusion being drawn *from* the two premisses through or in accordance with the principle which states that the two premisses are *alone* sufficient to secure validity for the conclusion. The principle cannot therefore be added co-ordinately to the premisses without contradiction. Moreover the above-formulated principle (which expresses the transitive property of the relation of equality) cannot be subsumed under the syllogistic principle. In the same way the syllogistic or inductive principle may be called a redundant or super-major, because it introduces a misleading or dispensable employment of the applicative principle.

§ 11. There is a special purpose in taking the inductive and syllogistic principles in illustration of super-majors, for many logicians have maintained that any specific inductive inference does not rest on an independent principle, but upon the syllogistic principle itself; in other words, they have taken syllogism to exhibit the sole form of valid inference, to which any other inferential processes are subordinate. Now it is true that the inductive principle could be put at the head of any specific inductive inference, and thus be related to the specific conclusion as the major premiss of a syllogism is related to its conclusion; but the same could be said of the syllogistic principle: namely that it could be put at the head of any specific syllogistic inference to which it is related in the same way as the major premiss of a syllogism is related to its conclusion. But, if we are further to *justify* the specific inductive inference by introducing the inductive principle, then,

by parity of reasoning, we should have to introduce the syllogistic principle further to justify the specific syllogistic inference. But in the case of the syllogism this would lead to an infinite regress as the following illustration will show. Thus, taking again as a specific syllogism, that

from (*p*) 'All labiates are square-stalked'

and (*q*) 'The dead-nettle is a labiate'

we may infer (*r*) 'The dead-nettle is square-stalked,'

and, adding to this as super-major the syllogistic principle, namely (*a*), we have the following argument:

(*a*) For every case of *M*, of *S* and of *P*: the inference 'every *M* is *P*, and *S* is *M*,  $\therefore$  *S* is *P*' is valid.

(*b*) The above specific syllogism is a case of (*a*).

(*c*)  $\therefore$  The specific syllogism is valid.

But here, in inferring from (*a*) and (*b*) together to (*c*), we are employing the syllogistic principle, which must stand therefore as a super-major to the inference from (*a*) and (*b*) together to (*c*), and therefore as super-super-major to the specific inference from *p* and *q* to *r*. This would obviously lead to an infinite regress.

We may show that a similar infinite regress would be involved if we introduced, as super-major, the inductive principle, by the following illustration. Taking again as a specific inductive inference that from 'All examined swans are white' we may infer with a higher or lower degree of probability that 'All swans are white'; and adding to this as super-major the inductive principle, namely (*a*), we have the following argument:

(*a*) For every case of *M* and of *P*: from 'every examined *M* is *P*,' we may infer, with a higher or lower degree of probability, that 'every *M* is *P*';

(b) The above specific induction is a case of (a).

(c)  $\therefore$  The specific induction is valid.

But, here we may argue in regard to this (a), (b), (c) as in the case of the previous (a), (b), (c). Thus, by introducing the inductive principle as a redundant major premiss, we shall be led as before, by an infinite regress, to a repeated employment of the *sylogistic* principle.

This whole discussion forces us to regard the inductive and sylogistic principles as independent of one another, the former not being capable of subordination to the latter; for we cannot in any way deduce the inductive principle from the sylogistic principle. Those who have regarded the sylogistic principle as ultimately supreme, have in fact arrived at this conclusion by noting that, as shown above, the inductive principle could be introduced as a major for any specific inductive inference, in which case the inference would assume the sylogistic form (a), (b), (c). But this in no way affects the supremacy of the inductive principle as independent of the sylogistic.

## CHAPTER II

### THE RELATIONS OF SUB-ORDINATION AND CO-ORDINATION AMONGST PROPOSITIONS OF DIFFERENT TYPES

§ 1. IN the previous chapter we have shown that the syllogism which establishes material conclusions from material premisses involves the alternate use of the Applicative and Implicative principles. Now these two principles, which control the procedure of deduction in its widest application, are required not only for material inferences, but also for the process of establishing the formulae that constitute the body of logically certified theorems. All these formulae are derived from certain intuitively evident axioms which may be explicitly enumerated. It will be found that the procedure of deducing further formulae from these axioms requires only the use of the Applicative and Implicative principles; these, therefore, cover a wider range than that of mere syllogism. But a final question remains, as to how the formal axioms are themselves established in their universal form. By most formal logicians it is assumed that these axioms are presented immediately as self-evident in their absolutely universal form; but such a process of intuition as is thereby assumed is really the result of a certain development of the reasoning powers. Prior to such development, I hold that there is a species of *induction* involved in grasping axioms in their absolute generality and in conceiving of form as



constant in the infinite multiplicity of its possible applications. We therefore conclude that behind the axioms there are involved certain supreme principles which bear to the Applicative and Implicative principles the same relation as induction in general bears to deduction; and, even more precisely, that these two new principles may be regarded as inverse to the Applicative and Implicative principles respectively. This being so, it will be convenient to denominate them respectively, Counter-applicative and Counter-implicative. It should be pointed out that whereas the Applicative and Implicative principles hold for material as well as formal inferential procedure, the Counter-principles are used for the establishment of the primitive axioms themselves upon which the formal system is based. We will then proceed to formulate the Counter-principles, each in immediate connection with its corresponding direct principle.

§ 2. The Applicative principle is that which justifies the procedure of passing from the asserting of a predication about 'every' to the asserting of the same predication about 'any given.' Corresponding to this, the Counter-applicative principle may be formulated:

'When we are justified in passing from the asserting of a predication about some *one* given to the asserting of the same predication about some other, then we are also justified in asserting the same predication about *every*.'

Roughly the Applicative principle justifies inference from 'every' to 'any,' and the Counter-applicative justifies inference from 'any' to 'every'; but whereas the former principle can be applied universally, the latter holds only in certain narrowly limited cases; and,

in particular, for the establishment of the primitive formulae of Logic. These cases may be described as those in which we see the universal in the particular, and this kind of inference will be called 'intuitive induction,' because it is that species of generalisation in which we intuit the truth of a universal proposition in the very act of intuiting the truth of a single instance<sup>1</sup>. Since intuitive induction is of course not possible in every case of generalisation, we have implied in our formulation of the principle that the passing from 'any' to 'every' is justified only when the passing from 'any one' to 'any other' is justified. Now there *are* forms of inference in which we can pass immediately from any one given case to any other; if it were not so, the principle would be empty. For instance, we may illustrate the Applicative principle by taking the formula: 'For every value of  $p$  and of  $q$ , " $p$  and  $q$ " would imply " $p$ ";' from which we should infer that 'thunder and lightning' would imply 'thunder.' If now we enquire how we are justified in asserting that for every value of  $p$  and of  $q$ , ' $p$  and  $q$ ' would imply ' $p$ ,' the answer will supply an illustration of the Counter-applicative principle. Thus, in asserting that "'thunder and lightning" would imply "thunder"' we see that we could proceed to assert that "'blue and hard" would imply "blue",' and in the same act, that "' $p$  and  $q$ " would imply " $p$ " for all values of  $p$  and of  $q$ .'

§ 3. The second inverse principle to be considered is the Counter-implicative. Before discussing this inverse principle, it will be necessary to examine closely the

<sup>1</sup> This is a special case of 'intuitive induction,' the more general uses of which will be examined in Chapter VIII.

Implicative principle itself, which may be provisionally formulated: 'Given that a certain proposition would formally imply a certain other proposition, we can validly proceed to infer the latter from the former.' Now we find that the one positive element in the notion of formal implication is its equivalence to potentially valid inference, and that there is no single relation properly called *the* relation of implication. We must therefore bring out the precise significance of the Implicative principle by the following reformulation: 'There are certain specifiable relations such that, when one or other of these subsists between two propositions, we may validly infer the one from the other.' From the enunciation of this principle we can pass immediately to the enunciation of its inverse—the Counter-implicative principle:

'When we have inferred, with a consciousness of validity, some proposition from some given premiss or premisses, then we are in a position to realise the specific form of relation that subsists between premiss and conclusion upon which the felt validity of the inference depends.'

Here, as in the case of the Counter-applicative principle, we must point out that there *are* cases in which we intuitively recognise the validity of inferring some concrete conclusion from a concrete premiss, before having recognised the special type of relation of premiss to conclusion which renders the specific inference valid; otherwise the Counter-implicative principle would be empty. In illustration, we will trace back some accepted relation of premiss to conclusion, upon which the validity of inferring the one from the other depends; and this

will entail reference to a preliminary procedure in accordance with the Counter-applicative principle; for every logical formula is implicitly universal. Thus we might infer, with a sense of validity from the information 'Some Mongols are Europeans' and from this datum *alone*, the conclusion 'Some Europeans are Mongols.' We proceed next in accordance with the Counter-applicative principle to the generalisation that the inference from 'Some *M* is *P*' to 'Some *P* is *M*' is always valid. Finally we are led, in accordance with the Counter-implicative principle, to the conclusion that it is the relation of 'converse particular affirmatives' that renders the inference from 'Some *M* is *P*' to 'Some *P* is *M*' valid.

§ 4. We have regarded the intuition underlying the Counter-applicative principle as an instance of 'seeing the universal in the particular'; and correspondingly the intuition underlying the Counter-implicative principle may be regarded as an instance of 'abstracting a common form in diverse matter.' But the *direct* types of intuition operate over a much wider field than the Counter-applicative and Counter-implicative principles: for, whereas the twin inverse principles operate only in the establishment of axioms, the *direct* types of intuition are involved wherever there is either universality or form. These direct types of intuition have been explicitly recognised by philosophers; but the still more purely intuitive nature of the procedure conducted in accordance with the twin inverse principles accounts for the fact that these principles have hitherto not been formulated by logicians. Moreover the point of view from which the inverse principles have been described

and analysed is purely *epistemic*; and the epistemic aspect of logical problems has generally been ignored or explicitly rejected by logicians. It follows also from their epistemic character that these principles, unlike the Applicative and Implicative principles of inference, cannot be formulated with the precision required for a purely mechanical or blind application.

§5. The operation of these four supreme principles is best exhibited by means of a scheme which comprises propositions of every type in their relations of super-, sub-, or co-ordination to one another. We propose, therefore, to devote the remainder of this chapter to the construction and elucidation of such a scheme.

I. *Superordinate Principles of Inference.*

Ia. The Counter-applicative and Counter-implicative.

Ib. The Applicative and Implicative.

II. *Formulae*: i.e. formally certified propositions expressible in terms of variables having general application.

IIa. Primitive formulae (or axioms) derived directly from IIIa in accordance with Ia.

IIb. Formulae successively derived from IIa by means of Ib.

III. *Formally Certified Propositions expressed in terms having fixed application.*

IIIa. Those from which IIa are derived by use of the principles Ia.

IIIb. Those which are derived from IIb by use of the Applicative principle Ib.

IV. *Experientially Certified Propositions.*

IVa. Data directly certified in experience.

IVb. Concrete conclusions inferred from IVa by means of implications of the type III, and therefore established in accordance with the Implicative principle, Ib.

I. The highest type consists of those principles under one or other of which every inference is subordinated. These superordinate principles consist of Ia: the Counter-applicative and Counter-implicative, to which *intuitional* inferences are subordinated; and of Ib: the Applicative and Implicative, to which *demonstrative* inferences are subordinated. Ia are those principles in accordance with which the primitive formulae (or axioms) of Logic are established. But the choice of logical formulae that are accounted primitive is (within limits) arbitrary, and since any comparatively self-evident logical formula, instead of being exhibited as derivative, could be regarded as established directly in accordance with these inverse principles, their scope must not be restricted to the establishment of the more or less arbitrarily selected axioms. It will be found later, when we discuss the types of proposition in level III, that the content or material upon which the inverse principles Ia operate, is supplied by the propositions of type IIIa. On the other hand, the Applicative and Implicative principles Ib stand in the relation of immediate superordination to the processes of inference by which from IIa are derived IIb, viz. the general formulae of deduction, induction, demonstration, probability, etc.

II. The characteristic common to all the propositions on the second level is that they are formally certified, and are expressible in terms of variable symbols. They are theoretically infinite in number, and may be divided

into two groups, 'primitive' and 'derivative'; but, as pointed out above, the line of demarcation between the two cannot be sharply drawn. Thus  $IIa$  comprises a small number of primitive formulae which are directly established in accordance with the twin inverse principles  $Ia$ : for example, the commutative and associative laws, the laws of identity and of negation, the modus ponendo tollens, etc., or such of these as have been selected as primitive. Next,  $IIb$  comprises an indefinite number of formulae successively derived from the primitive formulae  $IIa$ : for example, the dictum of the syllogism, and other more complicated logical formulae, as well as the rules of arithmetic and algebra. All the formulae of level II are implicitly universal in form; and most of those that are logical (as distinct from mathematical) assert relations of implication. Each formula in  $IIb$  is derived from previously certified formulae, and ultimately from those in  $IIa$ , the process of derivation being marked at each step by the relation 'therefore.' Now wherever a previously certified relation of implication is used for deriving a new formula (in which case its implicans must also have been previously certified in order that its implicate may be derivatively certified) the procedure is conducted in accordance with the implicative principle, to which therefore all such cases of inference are to be subordinated. Again, the process of successive derivation of the formulae of  $IIb$  entails explicit recognition of the implicit universality of the formulae from which they are derived; and this allows us, by means of the Applicative principle, to replace the illustrative symbols occurring in an earlier formula by any other symbols, in order to derive a new formula.

III. The third level contains formally certified propositions expressed entirely in linguistic terms of fixed application; and, like its predecessors, is to be divided into two sections, the division being made on precisely the same grounds as that between  $IIa$  and  $IIb$ . Thus the propositions of  $IIIa$  constitute the intuited material for deriving  $IIa$  in accordance with the inverse principles  $Ia$ ; and the propositions of  $IIIb$  are exhibited as derived from  $IIb$  in accordance with the applicative principle  $Ib^1$ . It will be seen, however, that the relation of  $IIIa$  to  $IIIb$  differs from that of  $IIa$  to  $IIb$  in that the two parts of III are not inferentially connected, as are those of II. The propositions comprised in  $IIIb$  are obtained from  $IIb$  by substituting words with fixed application for the variable symbols; these propositions, then, are specialised instances of the general formulae which constitute the second level, and are established from them in accordance with the applicative principle alone. Any logical text-book teems with examples of this procedure, where instances under such formulae as the modus tollendo tollens, or the syllogistic dictum are represented in words with fixed application, and then exhibited as derived (in accordance with the applicative principle) from the appropriate general formula. It is usual in these cases, however, to exhibit the conclusion as being *inferred* from the premisses, thus leading the reader to suppose that it is the conclusion which has been formally certified, whereas, properly speaking, what has been formally certified is the *relation of implication* of premisses to

<sup>1</sup> Hence the point of division between  $IIIa$  and  $IIIb$  cannot be precisely indicated.

conclusion. It will be found below that this distinction between implication and inference is the essential consideration in comparing III*b* with IV*b*.

IV. The fourth and lowest level consists of experientially certified propositions expressed in concrete terms; and again this level must be divided into two sections, viz. IV*a* the primitives and IV*b* the derivatives, these two sections standing in a relation to one another which in every respect agrees with the relation of II*a* to II*b*. Thus the propositions comprised in IV*b* are successively derived from experiential propositions that have been previously certified, and ultimately derived from the primitive experiential data which constitutes IV*a*. And again, as in the case of formally certified propositions, here, in that of experientially certified propositions, the point of division between the primitives and derivatives is not precisely fixed; the primitives of IV, like those of II, are supposed to be intuitively accepted, i.e. in this case perceptually guaranteed; but philosophers do not agree on the question of the kind and range of experiences that can be regarded as in this case immediate. Moreover, as regards experiential propositions admittedly derivative and not primitive, no logician or philosopher has as yet been able to show how they can be exhibited as derived ultimately from absolutely primitive data of experience. Hence, in expounding the logical nature of the propositions in this lowest level, attention must be chiefly directed to the mode in which any admittedly derived proposition is inferred from some previously certified proposition, without enquiring too closely as to the mode in which the previous certification had

been conducted, or whether this certification could properly be called perceptually immediate. The mode of deriving an experiential conclusion from experientially certified premisses may be explained quite briefly; the former is derived *directly* from the latter by means of some implication of type III, of which the implicans is composed of the previously certified premisses and the implicate is the conclusion required. Since in this process a relation of implication is transformed into the relation 'therefore,' it is obvious that the implicative principle alone is employed. But, to complete the exposition, we must trace the process of derivation one stage further back, namely to the general formulae of line II. Thus, while any conclusion in IV*b* is *directly* derived from premisses IV*a* by means of an implicative proposition of the type III, and so far employs the implicative principle alone; yet, since any proposition of type III is itself derived from some formula of type II in accordance with the applicative principle alone, it follows that both these principles are jointly involved in deriving experiential conclusions from experiential data. This mode of derivation is illustrated in any text-book example of a concrete syllogism, where from previous experiential certification of the premisses we infer the experiential certification of the conclusion. For the sake of variety we will choose, for illustrating the processes of deriving any conclusion IV*b*, the formula of pure induction, which, as was maintained in the preceding chapter, must be included amongst the formulae constituting the second level. Take for instance as premiss: 'Every examined case of an acquired characteristic is non-transmitted.'



This datum is regarded not, of course, as a mere summary of directly given experiences, but as the product of various constructive and inferential processes which may be supposed ultimately to be based on sense-data. Now by means of the concrete implication that 'every examined case of an acquired characteristic being non-transmitted would imply, with a certain degree of probability, that no acquired characteristic is transmissible,' conjoined with the certified fact that 'in all examined cases acquired characteristics are non-transmitted,' we infer the conclusion that 'with a certain degree of probability no acquired characteristic is transmissible.'

In this fourth line, we are representing propositions as proved, or as validly asserted on the basis of experiential knowledge, and this suggests an ambiguity in the use of the term 'ground' which is sometimes applied in philosophy to the experiential data which may be said to be *co-ordinate* with the experiential conclusions; the same term 'ground' being also applied to the logical formulae of induction or deduction which are *superordinate* to the experiential data and conclusions. This ambiguity in the use of the term is removed by thus recognising the distinction between superordinate and co-ordinate.

§ 6. In further elucidation of the scheme, we will show what exactly is involved in level II, where emphasis has been put upon the variable symbols. In logical text-books we find that an inference or implication is expressed in terms of variable symbols, such as *S*, *M*, and *P*, and this is always supplemented by a formula expressed entirely linguistically, but which is its mere equivalent. For example, it may be first

asserted that 'Every *P* is *Q* would imply that some *Q* is *P*'; and here the assertion of implication is understood as being implicitly universal, i.e. that it holds for all values of *P* and *Q*. This is usually supplemented by the so-called 'Rule for the Conversion of *A*,' viz. that 'Any universal affirmative proposition would imply the particular affirmative obtained by interchanging subject and predicate terms.' But this is merely an alternative formulation, and is not related to the former as a universal to its instance. We see therefore that the formulae of level II are not necessarily expressed in terms of variables, but may be expressed with precise equivalence in linguistic terms only. The possibility of this linguistic formulation depends upon the invention of a technical terminology which employs such terms as subject, predicate, conversion, universal, proposition, etc. The reason why what is called symbolic logic requires the employment of variable symbols is essentially because the logical formulae which it establishes are so complicated that a terminology could hardly be invented for dealing with them. There is therefore no difference of principle involved in the employment of variable symbols by symbolic logic and the employment of technical linguistic terms by ordinary logic. By the employment of the technical terminology of logic the variables entering into any formula are eliminated *en bloc*, leaving the formula with the same range of universality as before. In contrast with this, a proposition of level III, being obtained from level II by replacing *each* of the several variables by a particular word of fixed application, constitutes a single instance of the general formula. For instance, 'that every



trespasser will be prosecuted would imply that some prosecuted person is a trespasser<sup>1</sup>, is a specific assertion obtained by the applicative principle from the universal formula of conversion adduced above.

This last discussion of the distinction and connection between the use of variable symbols and that of linguistic terminology, points to certain respects in which the methods of symbolic logic differ, and others in which they agree with those of ordinary logic—a topic which will be treated at greater length in the following chapter.

<sup>1</sup> This illustration is chosen in order incidentally to suggest that the text-books are not always infallible, the form of implication in question being at least dubious.

## CHAPTER III

### SYMBOLISM AND FUNCTIONS

§ 1. THE value of symbolism, as is universally recognised, is due to the extreme precision which its employment affords to the process of logical demonstration. As a language it differs from all ordinary languages in three respects, viz. systematisation, brevity and exactness; and in these respects differs from all other languages in a way in which they do not differ from one another.

Now, when we examine the language of symbolism, we find that symbols are of two fundamentally distinct kinds, which I propose to call *illustrative* and *shorthand*. In such familiar logical forms as 'S is P,' 'Every M is P,' etc., S, M, P, exemplify illustrative symbols. Thus an illustrative symbol is represented by a single letter chosen from some alphabet. Shorthand symbols, on the other hand, are mere substitutes for words, and serve the obvious purpose of saving time in reading and space in writing. Some of them, in fact, are literal abbreviations, such as 'rel.' for 'relative,' 'prop.' for 'proposition,' 'indiv.' for 'individual.' Others again are arbitrarily shaped marks standing for simple words such as *not*, *and*, *or*, *if*, *is*, *identical with*. A third kind of shorthand symbol is one introduced in the course of a symbolic calculus, and defined in terms of combinations of other shorthand symbols, and ultimately in terms of the simple symbols introduced at the outset. So far,

a shorthand symbol has all the characteristics of a word or a word-complex—only differing from these in satisfying the essential symbolic requirements of systematisation, brevity and exactness. In one respect, however, these symbols differ from such word-complexes as 'that man,' 'the river,' 'Mr Smith,' 'this experience,' 'my present purpose,' in that these latter have a meaning or application—not universally fixed—but determined only by means of context; whereas the symbols of Logic have an unalterable meaning wholly independent of context, and resemble rather, such word-complexes as 'rational animal,' 'loud,' 'hard,' 'church,' differing from these however in being strictly unambiguous. Ordinary Logic generally dispenses with symbols of this kind—the most familiar exception being Dr Keynes's '*SaP*,' which is shorthand for 'Every *S* is *P*,' etc. On the other hand, Dr Keynes himself shows, in his Appendix C, how certain complicated problems, previously relegated to Symbolic Logic, can be solved without recourse to shorthand symbols, illustrative symbols only being introduced.

Now an important character of the shorthand symbol is that its constancy is logical or formal and not experiential or material. A formal constant is one whose meaning is to be understood by the logician as such; that is to say, logic pronounces it either as indefinable—because understood without requiring definition—or as definable in terms of logically understood constants alone. The following is a rough classification of formal constants expressed in ordinary language: (1) the articles or applicatives; *a*, *the*, *some*, etc. (2) the negative *not*; and the conjunctions *and*, *or*, *if*, etc. (3) the copula *is*;

and certain prepositions such as *of*, *to*, in some of their meanings. (4) certain relations such as *identical with*, *comprised in*. (5) such modal adjectives as *true*, *false*, *probable*, etc. Formal constants are to be contrasted with material in that the meanings of the latter are to be understood in terms of ideas or conceptions outside the sphere of logic. The division between formal and material constants, i.e. between what is and what is not required for the understanding of logical principles, can ultimately be rendered precise only after a complete logical system has been constructed. For instance, numerical adjectives such as *two* and *five* would have been pronounced as merely material at the stage at which the logical system had not been carried on into its mathematical developments. Ideas that are immediately recognised as material relatively to the essentials of logic are those of sense-qualities, or of the properties and characteristics of physical and mental entities. Temporal and spatial relations, being in one aspect subsumable under the conceptions of order, would, so far, be called formal or logical, but, inasmuch as these relations 'actually have a specific—over and above their generic—significance, they must be treated also as having an experiential or material source. The same holds of the determinates of a determinable, inasmuch as experience is required in order to present to the mind any single determinable and to distinguish one determinable from another, whilst the discussion of the formal relations of incompatibility, order, etc., between determinates under any determinable is purely logical.

Since shorthand symbols and the words or word-complexes of ordinary language function in the same

way, there is no essential difference between them if we take the symbols or words in isolation apart from consideration of the mental processes involved in their use. The psychological distinction is—not between words and symbols as such—but between the linguistic and the symbolic mode in which we think with their assistance. Thus, in linguistic thought, the words or symbols presented in imagination or vocalisation are the means or instruments by which we can attend to or think about the objects for which they stand. On the other hand, such a phrase as 'Waterloo was fought in 1815' might illustrate the symbolic use of language which consists—not in thinking about the objects for which the words stand—but in mentally rehearsing the language in which propositions previously accepted have been expressed. Now the previous acceptance of these propositions must have entailed genuine processes of thinking; but, when they are recalled, we need not repeat these mental processes. It is in this way that the symbolic is distinguished from the linguistic use of words or symbols. In the latter, we are thinking by the use of words; whereas, in the former, recall of the words serves merely as a substitute for a previous act of thought<sup>1</sup>.

§ 2. These preliminary considerations bring us to the question: What actually happens in the mind of the symbolist, when he is either constructing or intelligently following the formulae of a symbolic calculus? In the first place, the axioms of the calculus can only be established by the use of what I have called the Counter-

<sup>1</sup> This subject will be found to be more fully treated in Dr Stout's *Analytic Psychology*.

applicative and Counter-implicative principles, and here genuine thought is required on the part of the symbolist. In the second place, the construction of any symbolic calculus involves the procedure of inference; and this is conducted always in accordance with the Applicative principle, and, in the case of the logical calculus, also in accordance with the Implicative principle. When proceeding in accordance with these principles, the symbolist is actually thinking; he is not merely recalling verbal formulae in which the results of previous acts of thought have been expressed. In the third place, even a perfectly constructed symbolic system would need to introduce some axioms, as also some propositions derived from axioms, that can only be expressed in non-symbolic terms. This necessary recourse to ordinary language in developing a deductive system shows that direct attention to meanings, presented linguistically, is entailed in the intelligent following of even a professedly symbolic exposition. Lastly, the extent to which thought can be dispensed with, when working a calculus, depends very largely and essentially upon the extent to which the system requires what may be called interpretation clauses such as 'when  $P$  stands for any proposition,' or 'where  $x$  is to be understood as a variable and  $a$  as a constant.' If the symbolic language is so constructed that a minimum of interpretation clauses is required, then there is a corresponding minimum in the extent to which actual thinking is involved. But, however few interpretation clauses are required, the intelligent use of symbolic formulae cannot be reduced to a merely mechanical process. This will be still more apparent from an examination of the nature of a symbolic system in which both

shorthand and illustrative symbols enter in combination with one another.

§ 3. For this purpose we will further consider the characteristics of illustrative symbols. These, being nothing but arbitrarily chosen letters of the alphabet, differ from words of ordinary language in that they cannot be interpreted as standing for this rather than for that specific object or idea; and hence, in the nature of the case, have a variable application. The writer or reader of a symbolic system must always bear in mind, however, that the variability in application of an illustrative symbol in any given case is not wholly unrestricted, but is limited within an understood range. Thus a single letter used illustratively must be understood to be restricted in one case (say) to any substantive; in another (say) to any adjective; and in another again (say) to any proposition,—these being the three most prominent categories to which illustrative symbols are applied. Symbolic devices may, indeed, be invented by which to distinguish one kind of symbol as applicable to a substantive, and another kind to some other specific category; but the range of application to be understood by letters taken in combination could not be indicated by any such device. When single letters are bound together into a complex by means of logical constants, then a further act of intelligence is required in interpreting such complex. For example, understanding in the first place the letters  $p$ ,  $q$ ,  $r$ , to stand for propositions, such constructs as ' $p$  and  $q$ ,' ' $p$  or  $q$ ,' ' $p$  if  $q$ ,' must be *further* interpreted as also constituting propositions. Thus, when a formula about any or all propositions has been established, we may proceed to

apply it to any complex such as ' $p$  and  $q$ ' or ' $p$  if  $q$ ' and so on in accordance with the Applicative principle, inasmuch as each of such complexes constitutes a proposition. Similarly, when such letters as  $x$ ,  $s$ ,  $t$  are understood to stand for substantives, and such letters as  $p$ ,  $q$ ,  $r$  to stand for adjectives, then a further act of intelligence is required to interpret such a complex as ' $s$  is  $p$ ' as standing for a proposition. This presupposes that the logical analysis of the simple proposition into the form ' $s$  is  $p$ ,' where  $s$  is understood to stand for a substantive and  $p$  for an adjective, has been discussed and established in a preliminary account in which words and *not* symbols were employed. Propositional significance having been attached to this form of construct, a distinct act of intelligence is required when, in uniting say ' $s$  is  $p$ ' with ' $t$  is  $q$ ' in some form of combination, the resulting construct is understood to stand for a proposition. As another example illustrating the need for intelligent activity in symbolic work, we may take the two propositional forms ' $s$   $e$   $p$ ' and ' $x$   $i$   $y$ ,' where ' $e$ ' is shorthand for the copula '*is*' and ' $i$ ' for *is identical with*. Not only must these forms be interpreted as standing for propositions, but the relation for which ' $e$ ' stands must be understood to be different from that for which ' $i$ ' stands. In consequence, when these two forms occur, reference must be made to one set of established formulae for the one case, and to a different set for the other. The necessity for using this modicum of intelligence is to be contrasted with the purely blind or mechanical process required of the reader or writer in making use of the formulae to which he refers; for, in this latter process, he need attach no significance to ' $e$ '

or to 'i,' as each standing for its own specific relation. The examples adduced have been selected on the ground of their simplicity, but complex examples would have brought out more forcibly the importance of the distinction between the intelligent and the merely mechanical operations required in working a symbolic system.

§ 4. Now the variability that characterises illustrative symbols constitutes a special feature of symbolism, and its further discussion requires the introduction of the notion *function*. This term is used by logicians and mathematicians in a sense quite unconnected with the biological meaning of the term. The notion of a function is closely connected with the notion of a construct, but the former must be understood *relationally*, whereas this is not obviously the case with the term construct. Thus, we should speak of a certain construct as being a function *of* certain enumerated constituents. The notation for a function in general is  $f(a, b, c, \dots)$  where  $a, b, c, \dots$  stand for the constituents; and where the order in which these constituents are written is essential, so that  $f(a, b, c, \dots)$  is not necessarily equivalent to  $f(a, c, b, \dots)$ . Thus any function of  $a, b, c, \dots$  is a construct involving  $a, b, c, \dots$ . But, if this were all that could be said about a function, the term would have no special value, since it would be a mere synonym for 'construct involving.' The importance of the notion of function lies in the fact that we may speak of the *same* function in reference to *different* constituents, whereas the same construct would of course entail the same constituents. Thus, if  $C$  be a certain construct involving  $a, b, c, \dots$ , and if  $D$  be another construct involving  $p, q, r, \dots$ , then  $C$  is said to be the same

function of  $a, b, c, \dots$  as is  $D$  of  $p, q, r, \dots$ , when the substitution of  $p$  for  $a, q$  for  $b, r$  for  $c$ , etc., would render  $D$  identical with  $C$ . Thus, in order to decide as regards two constructs, whether they express the same or a different function, we must specify the constituents of which the construct is regarded as a function; and, to avoid all possible ambiguity, *all* the constituents for which substitutions have to be made must be enumerated. To explain this necessity, it must be pointed out that a construct may involve, implicitly or explicitly, other constituents in addition to those *of* which it is to be regarded as a function. In order to indicate the sameness of function exhibited by different constructs, it is therefore essential to enumerate those constituents for which substitutions are contemplated. These constituents will be called *variants*<sup>1</sup>, because it is these and these alone that have to be varied in order to obtain the *different* constructs that exhibit the *same* function. On the other hand, in exhibiting identity of function, terms entering into the construct that are not to be replaced by some other terms will be called constants or non-variants. Hence the distinction between a variant and a non-variant constituent of a construct has relevance only to functional identity. Since a function and its variants are to be understood relationally to one another, we may speak of the variants *for* a certain function just as we speak of a function *of* certain

<sup>1</sup> The word *variant* is here and throughout used in place of the mathematically technical word *argument*, partly in order to prevent confusion with the ordinary logical use of the latter word, and partly in order to bring out the distinction and connection between the notion of *variant* and that of *variable*.



variants. In a complicated symbolic system it is found to be convenient to use, in place of a singular or proper name, an illustrative symbol—which, *quâ* symbol, must be what is called *variable*. Variability is therefore the mark of an illustrative symbol as such, whereas the contrast between variant and non-variant holds—not of a symbol—but of that for which the symbol may stand; and, as has been said, this latter contrast has no significance apart from the notion of a function.

§ 5. In considering the constituents of a construct with a view of indicating which are to be variants and which non-variants for a function, we must first note the distinction between material and formal constituents. Now as regards the strictly formal constituents of a construct, logic never contemplates making substitutions for these; hence, in all applications of the notion of a function in reference to its variants, two cases only have to be considered; (1) the function for which *all* the material constituents are treated as variants, and (2) the function for which *some* of the material constituents are treated as constants and others as variants—in both cases the formal constituents being understood to be constants. When (1) all the material constituents are to be varied, then the function may be said to be formal; and the *form* of a construct is a brief synonym for the formal function which it exhibits. But, when (2) some of the material constituents are to be constant, then the function will be said to be non-formal. It follows that, when two constructs can be said to exhibit the same formal function, their reduction to identity is effected by taking all the formal constituents to be constant, and replacing all the material

constituents of the one by those of the other. But, when two constructs are said to exhibit the same non-formal function, their reduction to identity is effected by taking certain of the material, as well as all the formal, constituents to be constant, and replacing all the remaining material constituents of the one by those of the other. A formal function is a function of all the material constituents, since all these are to be varied; but a non-formal function is a function of only some of the material constituents, because only some of these are to be varied.

We may take the following as illustrations of formal functions: The construct 'a good boy' is the same function of the variants *good* and *boy* as is 'a difficult problem' of the variants *difficult* and *problem*; 'Socrates is wise' is the same function of *Socrates* and *wise*, as is 'London is populous' of *London* and *populous*; 'red or heavy' is the same function of *red* and *heavy* as is 'loud or pleasant' of *loud* and *pleasant*. We may compare these simple examples with similarly simple examples in arithmetic. The arithmetical construct 'three days plus seven days' is the same function of the two variants *three days* and *seven days* as is 'five feet plus four feet' of the two variants *five feet* and *four feet*; 'four days multiplied by three' is the same function of *four days* and *three* as is 'seven feet multiplied by two' of *seven feet* and *two*, etc. These illustrate formal functions because the only constituents which are constant are formal: namely 'a,' 'is,' 'or,' 'plus,' 'multiplied by,' respectively. Each of the above examples exhibits a *specific* formal function, and serves to explain the *general* notion of a formal function. We



may take similar examples to illustrate the general notion of a non-formal function. Thus taking *boy* as constant, 'a good boy' is the same function of *good* as is 'a tall boy' of *tall*; taking *good* as constant, 'a good boy' is the same function of *boy* as is 'a good action' of *action*; taking *pleasant* as constant, 'loud or pleasant' is the same function of *loud* as is 'bright or pleasant' of *bright*; taking *wise* as constant, 'Socrates is wise' is the same function of *Socrates* as is 'Plato is wise' of *Plato*; taking *Socrates* as constant, 'Socrates is wise' is the same function of *wise* as is 'Socrates is poor' of *poor*, etc., etc. And in general the specific function exhibited by a given construct varies according to the constituents of the construct that operate as variants<sup>1</sup>.

<sup>1</sup> It will be observed that in the above illustrations of non-formal functions we have used adjectives and substantives indifferently as constants or as variables. Now in Mr Russell's first introduction of the notion of function, he appears to limit the application of the notion to the case where the substantive is variable and the adjective is constant. It is true that he extends the notion to include the cases in which the reverse holds; yet throughout he adopts an absolute distinction between the two constituents of a proposition which I have called substantive and adjective, inasmuch as he treats the substantive as the typical kind of entity which can stand by itself, the adjective never being allowed to stand by itself. Thus I am repeating his illustration in giving 'Socrates is wise' as the same function of *Socrates* as is 'Plato is wise' of *Plato*, since here the substantive terms *Socrates* and *Plato* are allowed to stand by themselves. But the parallel example, that 'Socrates is wise' is the same function of *wise* as is 'Socrates is poor' of *poor*, is not recognised by Mr Russell, because he does not allow such adjective-terms as 'wise' and 'poor' to stand by themselves. The consequences of this contrast, which I hold to be fundamentally fallacious, between the substantive and the adjective as constituents of a proposition, infect the whole of his logical

§ 6. A classification has been given, in an earlier section, of those formal constituents of a construct that are expressible *in words* or in shorthand symbols understood as equivalent to words. Such formal constituents may be called *explicit* in distinction from others which are more or less latent and not usually expressed in words. Reserving the name 'constituent,' for the material variants, and 'formal component' for those formal constants that are explicitly expressed, the implicit formal constants may be conveniently termed 'elements of form.' Of these, several different kinds are to be distinguished:

(1) *Ties*. These are more or less latent elements of form, inasmuch as it is a matter of accident whether they are expressed by some separate word or by some form of grammatical inflection.

(2) *Brackets*. A construct may be composed of sub-constructs, and these again of sub-sub-constructs

system. Without entering into elaborate detail, it would be impossible fully to justify my difference from Mr Russell on this matter; but what I take to be perhaps the root of the error is that he treats the general notion of function before giving examples of the simplest functional forms upon which the more complicated functions are built. It is true that he illustrates a function by such an elementary example as '*x* is a man' where *x* stands indifferently for Socrates or Plato, etc., but he does not bring out the speciality of this form of proposition, which does in fact exhibit the specific function which is constructed by means of the copula 'is.' In mathematics the general notion of function is reached by building up constructs out of such elementary functions as those indicated by  $+$   $\times$   $-$  etc., but in Mr Russell's system it seems impossible to explain and reduce to systematic symbolisation the process by which any propositional function whatever is constructed.

I hope to treat more fully elsewhere this point of difference between Mr Russell's system and my own.

and so on, until we reach the ultimate constituents, namely those that are expressed, not as constructs, but as 'simples,' where by 'simple' is not meant incapable of analysis, but merely unanalysed. The operation of binding constituents into a unity to constitute a sub-construct I shall call *bracketing*. In speaking, the distribution of brackets is indicated by pauses or vocal inflections; and, in writing, by punctuation marks. But, as the employment of these signs is not governed by any systematic principle, they must be replaced in logical or mathematical symbolism by some conventional notation.

(3) *Connectedness*. Two sub-constructs will be called *unconnected* when one is a function of the simple terms  $a, b, c$  (say), and the other of the simple terms  $d, e$ —the terms of the one not recurring in the other. On the other hand, two sub-constructs will be called *connected* when one is a function of  $a, b, c$  (say), and the other of  $a, e$ —the term  $a$  recurring in the two. This distinction is of importance when we have to determine what constituents of a function can be taken as variants; for the several variants for a function must be independently variable, and in the case of any two complex constituents, if these are connected (in the sense explained), they cannot be made to vary independently the one of the other. Thus, in the above illustration of two sub-constructs that are respectively functions of  $a, b, c$  and of  $a, e$ , the variants for the function exhibited by the construct must be taken to be the 'simple' constituents  $a, b, c, e$ , and *not* the connected sub-constructs themselves. But, when a construct contains unconnected sub-constructs, as in the example of the

sub-constructs that are functions respectively of  $a, b, c$  and of  $d, e$ , then it may be regarded *either* as a function of the several ultimate terms involved in the different sub-constructs, namely  $a, b, c, d, e$ ; *or* alternatively, as a function of the sub-constructs themselves.

(4) *Categories*. Every material, and therefore variable, constituent belongs to a specific logical category or sub-category which is not usually expressed in words. Thus the proposition 'Socrates is wise' is understood as it stands without being expanded into the form 'The substantive Socrates is characterised by the adjective wise.' Nevertheless the formal significance of the proposition for the thinker depends upon his conceiving of 'Socrates' as belonging to the category *substantive*, and of 'wise' as belonging to the category *adjective*. These must therefore be included amongst the latent elements of form. It further follows from the recognition of this formal element, latent in every material constituent, that the range of variation for any material constituent is determined by the logical category—substantive, adjective, relational adjective, as the case may be—to which it belongs. In other words, the material constituents which may replace one another, in order that the construct may exhibit the same function in its varied exemplifications, must all belong to the same logical category or sub-category.

§ 7. This account of the formal elements of a construct leads to an examination of different types of function. Amongst the functions of logic the conjunctive and the predication are the most fundamental. A function is called *conjunctive* when the component that determines its form is the negative *not*

or some logical conjunction; and the variants for such a function are always, strictly speaking, propositions, as is also the construct itself. A function is called *predicational* when the component that determines its form is the characterising tie, which unites two variants related to one another as substantive to adjective. Thus there is only one elementary predicational function, namely the characterising function represented by the copula 'is'; whereas there are five elementary conjunctive functions represented respectively by the operators, 'not,' 'and,' 'if' and its converse, 'or,' 'not-both.' Just as a conjunctive function may exhibit any degree of complexity made up of these elementary conjunctive functions, so a predicational function may exhibit any degree of complexity made up of recurrences of the characterising function in sub-constructs and sub-sub-constructs, etc. An important distinction between these two types of function introduces the notion of functional homogeneity. A function is said to be *homogeneous* when all its variants belong to the same category as itself. Now, since a conjunctive function takes propositions as its variants and is itself a proposition, it illustrates a homogeneous function; but, since a predicational function constitutes a construct under the category proposition out of constituents under the respective categories, substantive and adjective, it illustrates a heterogeneous function. Under this head are also to be included secondary propositions which predicate adjectives of primary propositions, and propositions which predicate secondary adjectives of primary adjectives; for the subjects of these propositions are quasi-substantives, and the propositions themselves

are of a different order of category from their constituent terms.

§ 8. We will proceed to apply the notion of connectedness to these two types of function. A conjunctive function is a function of those propositional sub-constructs which are unconnected, but not of those which are connected with one another through identity of some of the terms involved. For such sub-constructs, though properly regarded as *constituents*, cannot be taken as *variants*, since they cannot be freely varied independently of one another. Thus the variants for a conjunctive function which is also connectional are not the connected sub-constructs themselves, but the ultimate propositions or 'simples' of which they are constituted; e.g. in the construct

$\{(p \text{ and } q) \text{ or } (p \text{ and } r)\} \text{ and } (x \text{ or } y)$

the constituents that may be taken as variants are  $p, q, r, (x \text{ or } y)$ ; and in the construct

$\{(p \text{ and } q) \text{ or } (p \text{ and } r)\} \text{ and } (q \text{ or } y)$

the only constituents that can be taken as variants are  $p, q, r, y$ . In these symbolic illustrations, the ultimate constituents are unanalysed propositions; but the same distinction between connected and unconnected sub-constructs holds for a conjunctive function of propositions that are expressed analytically in terms of subject and predicate. For example, ' $A$  is  $p$  or  $B$  is  $q$ ' illustrates a conjunctive function of the two unconnected sub-constructs ' $A$  is  $p$ ,' ' $B$  is  $q$ .' On the other hand, ' $A$  is  $p$  or  $A$  is  $q$ ' is not a function of the sub-constructs ' $A$  is  $p$ ,' ' $A$  is  $q$ ' because these are connected; but must be taken as a function of the three

ultimate constituents  $A, p, q$ . Again, ' $A$  is  $p$  or  $B$  is  $p$ ' is not a function of ' $A$  is  $p$ ,' ' $B$  is  $p$ ,' but of the ultimate constituents  $A, B, p$ . The connectedness in the former case is through identity of the substantive  $A$ ; in the latter through identity of the adjective  $p$ . Similar examples of connectedness occur, in which 'if' or 'not-both' or 'and' enter in the place of 'or.'

Ordinary language adopts abbreviated expressions for propositions that are connected, through identity of *subject*, by constructing a compound *predicate*, e.g. ' $A$  is  $p$  or  $q$ ,' ' $A$  is  $p$  and  $q$ '; as also for propositions that are connected, through identity of *predicate*, by constructing a compound *subject*, e.g. ' $A$  or  $B$  is  $p$ ,' ' $A$  and  $B$  are  $p$ .' This is extended to any number of terms enumeratively assigned for which language supplies us with a special condensed mode of expression. Thus the alternative function is condensed into the form: 'Some one or other of the enumerated items is  $p$ '; and the conjunctive function into the form: 'Every one of the enumerated items is  $p$ .' Such forms are usually restricted to enumerations of substantival items: for example, 'Some one of the apostles was a traitor,' 'Every one of the apostles was a Jew.' But it is possible to extend the form to enumerations of propositional or of adjectival items; for example, 'Some one of the axioms of Euclid is unnecessary for the purpose of establishing the theorems of geometry'; or 'Every one of the qualities characterising  $A, B, C$  characterises  $D$ .'

§ 9. A special notation has been adopted by the symbolists for representing such condensed expressions. In this notation, an illustrative symbol such as  $x$  enters as an *apparent* variable (to use Peano's phraseology);

by which is meant that the proposition in which  $x$  occurs—though it appears to be, yet is not in reality—about  $x$ , inasmuch as its content is not changed when any other symbol, say  $y$ , is substituted for  $x$ . The typical mode of formulating propositions on this principle is: 'Every item, say  $x$ , is  $p$ ,' or 'Some item, say  $x$ , is  $p$ ,' where it is obvious that the force of the proposition would be unaltered if we substituted  $s$ , or  $y$ , or  $z$ , for  $x$ . If  $X$  is the name of the class that comprises all such items as  $x$ , then the above forms are equivalent to 'Every  $X$  is  $p$ ,' and 'Some  $X$  is  $p$ ' respectively. The ultimate constituents of such universal or particular propositions are the simple propositions of the form ' $x$  is  $p$ ' which are conjunctively combined for the universal, and alternatively combined for the particular. The phrases 'Every  $X$ ,' 'Some  $X$ ,' therefore, though obviously constituents of the *sentence*, do not denote genuine constituents of the proposition of which the sentence is the verbal expression. Since then the constituents of the general proposition are singular propositions of the form ' $x$  is  $p$ ,' such a class-name as  $X$  and such a variable name as  $x$ , which are in danger of being identified, must be carefully distinguished. To the former the distributives *some* or *every* can be prefixed, never to the latter. [See Part I, Chapter VII.]

When we use a symbolic variable or illustrative symbol  $x$  to construct the proposition ' $x$  is  $p$ ' say,  $x$  stands, not for a class-name, but for a special kind of singular name, only differing from the ordinary singular name in that it stands indifferently for *any* substantive name, such as 'Socrates' or 'Cromwell' or 'this table' or 'yonder chair.' To bring out more pre-

cisely the distinction between a symbolic variable and a class-name, we may suppose that in a certain context *s* stands indifferently for any *person* such as 'Socrates' or 'Cromwell'; or again, indifferently for any *article of furniture* such as 'this table' or 'yonder chair.' Now 'person' and 'article of furniture' are class-names, and in the instances adduced the symbolic variable *s* stands—not for the class-name—but in fact for *any* singular name (proper or descriptive) that denotes an individual comprised in the class 'person' for the one case, and the class 'article of furniture' for the other case. What holds of a substantive-name *s* holds also of an adjective-name *p* or of a class-name *c*. Thus, in the form '*s* is *p*,' where '*is*' represents the characterising tie, *p* stands for any one indifferently assignable adjective comprised, say, in the class *colour*, but not for the class itself to which the distributives 'every' or 'some' can be prefixed. Again, in the form '*s* is comprised in *c*,' *c* represents a singular class-name standing for any one indifferently assignable class; and the limits of variation for the variable *c* could be expressed in terms of a class of a higher order comprising it. Thus the symbol *c* is equivalent to a variable proper class-name, and, like the substantive-name *s* and the adjective-name *p*, is to be contrasted with the class in which it is comprised. The names *substantive*, *adjective*, *proposition*, etc., which denote logical categories, i.e. the ultimate comprising classes, are not *variable* proper names, but names bearing fixed or constant significance, having so far the character of shorthand symbols in that they stand for logical constants, not for material variables. Thus the employ-

ment of the illustrative symbol as an apparent variable—i.e. to stand indifferently for any one or another object—makes possible the use of the same symbol, recurring in a given context, to stand for the same object. It thus fulfils the same function in a complex symbolic formula as the proper name in ordinary narrative, where the use of the pronoun in complicated cases would be ambiguous. The construction of such formulae requires the use, in a symbolic system, of apparent variables in place of class-names.

§ 10. We have seen that certain phrases containing, implicitly or explicitly, the conjunctions *and* or *or*, though linguistically intelligible, do not really represent genuine constructs. This raises a wider and more fundamental problem in regard to the nature of logical conjunctions when used in constructing a compound out of enumerated items. Can conjunctions serve to construct compound substantives or compound adjectives in the same way as they operate in constructing compound propositions? Now I shall maintain that while the nature of an adjective is such that we may properly construct a compound adjective out of 'simple' adjectives just as we may construct a compound proposition out of 'simple' propositions, yet the nature of any term functioning as a substantive is such that it is impossible to construct a genuine compound substantive. Thus 'rational and animated' represents a genuine conjunctive adjective, since it is equivalent in meaning to the simple adjective 'human'; and 'one or other of the colours approximating to red' is a genuine alternative adjective, since it is equivalent in meaning to the simple adjective 'reddish.' And again, more generally, where no single adjectival



word represents such a conjunction of adjectives as 'square and heavy,' 'red or green,' these are still to be regarded as genuine adjectival constructs on the ground that they agree in all essentially logical respects with simple adjectives, from which in fact they cannot be distinguished by any universal criterion. It follows, therefore, that no contradiction will ensue from replacing a simple by a compound adjective in any general formula holding of all adjectives as such. At the same time it must be pointed out, as regards alternative adjectival constructs, that no single or *determinate* adjective can be *identified with* such an alternative or indeterminate adjective as 'red or green,' 'one or other of the colours approximating to red.' In this respect, as we shall see, an alternative adjectival construct precisely resembles a substantival construct. Turning then to substantival constructs, it is obvious in the first place that a conjunctive enumeration of substantives such as 'Peter and James' or 'Every one of the apostles' does not represent any single or determinate man. It might, however, be maintained that such phrases represent a couple of men or a class of men, and that a couple or a class comprising substantives is itself of the nature of a substantive. Such a view would, however, involve a confusion between the enumerative and the conjunctive *and*. A statement about 'Peter and James' or 'Every one of the apostles' is really not about the compound construct that appears to be denoted by its subject-term, but must be analysed into a conjunctive compound of singular propositions. Thus in the statement 'Peter and James were fishermen' the subject-term uses *and* enumeratively. The conjunctive *and* can be shown

to enter only when we analyse the statement into the form 'Peter was a fisherman and James was a fisherman.' The case of an *alternative* enumeration of substantives, such as 'Peter or James' or 'Some one of the apostles,' is less obvious than that of a conjunctive enumeration of substantives. To prove that the alternative enumeration does not represent a genuine substantive, it will be convenient to take a proposition in which the enumeration occurs in the predicate. Thus 'Nathaniel is one of the apostles' or 'Bartholomew is one of the apostles' would appear to be expressible in the form 'Nathaniel is-identical-with one or other of the apostles' or 'Bartholomew is-identical-with one or other of the apostles.' But, if this is allowed, the conjunction of these two propositions would imply that 'Nathaniel is-identical-with Bartholomew,' since things that are identical with the same thing are identical with one another. Now that 'Nathaniel is-identical-with Bartholomew' may or may not be the case; but it certainly would not follow from the fact that Nathaniel was one of the apostles and that Bartholomew was one of the apostles. In order correctly to formulate the proposition 'Nathaniel was one of the apostles' in terms of the relation of identity, it must be rendered: 'Nathaniel is-identical-with Peter or identical-with Bartholomew or identical-with Thaddeus, etc.' In this form, the alternants are not the proper or substantival names Peter, Bartholomew, Thaddeus, etc., but the adjectival terms 'identical with Peter,' 'identical with Bartholomew,' 'identical with Thaddeus,' etc. These latter being recognised as adjectives, the reconstructed proposition assumes the form '*A* is *p* or *q* or *r*, etc.' where 'is'



represents the characterising tie, and  $p, q, r...$  stand for adjectives, so that (as alleged above) the new predicate expresses a genuine construct.

§ 11. A further and more general explanation may now be given of the principle according to which a proposition containing a fictitious construct must be reformulated. What holds of the relation of identity (as in the particular example concerning the apostles) holds of any relation whatever: that is to say, taking  $\hat{r}$  to stand for any relation, the phrase ' $\hat{r}$  to  $a$  or  $b$  or  $c...$ ' does not express a genuine construct and must be replaced by the phrase ' $\hat{r}$  to  $a$  or  $\hat{r}$  to  $b$  or  $\hat{r}$  to  $c...$ ' which is an alternative of adjectives. For example, the proposition 'This action will injure either Germany or England' must be transformed into 'This action will either injure Germany or injure England.' The essential points in this transformation can best be indicated with the help of vertical lines for brackets. Thus:

$$x \mid \text{ is } \hat{r} \text{ to } \mid a \text{ or } b \text{ or } c$$

is corrected into

$$x \mid \text{ is } \mid \hat{r} \text{ to } a \text{ or } \hat{r} \text{ to } b \text{ or } \hat{r} \text{ to } c.$$

In the former the two principal constituents of the proposition are linked by the relational predication 'is  $\hat{r}$  to,' in the latter by the characterising tie 'is.' In order that the predicate in the latter case should constitute a genuine construct, what is essential is, not that the subject term should stand for a substantive in any absolute sense, but only that it should function as a substantive relatively to the adjectival predicate; and it is the characterising tie which indicates this relative conception of substantive to adjective. Thus the term  $x$

may be either a substantive proper, an adjective or a proposition, and the same holds of the terms  $a, b, c$ , with which  $x$  is connected by the relation  $\hat{r}$ .

Examples may be given of propositions based upon the forms ' $x$  is  $\hat{r}$  to  $a$  or  $b$ ,' ' $x$  is  $\hat{r}$  to  $a$  and  $b$ ,' in order further to illustrate the principles under discussion.

(1) In the example just given: 'This action will injure either Germany or England,' which must be rendered 'This action will either injure England or injure Germany,' the terms  $x, a, b$ , are all substantives proper. But taking

(2) ' $p$  characterises either  $a$  or  $b$  or  $c$ ,' which has to be transformed into ' $p$  either characterises  $a$  or characterises  $b$  or characterises  $c$ ,' the subject term is an adjective and may be called primary relatively to the predicate terms which function as secondary adjectives<sup>1</sup>.

In (3): ' $A$  has asserted  $p$  or  $q$  or  $r$ ,' the subject term stands for a person (i.e. for a substantive proper), and the terms  $p, q, r$  in the predicate are propositions. Since here the terms alternatively combined are themselves propositions, the expression as it stands would be correct if its intention were to state that the compound proposition ' $p$  or  $q$  or  $r$ ' was asserted by  $A$ . But, if it were intended to state that one or other of the assertions  $p, q, r$  had been made by  $A$ , then (3) should be amended

<sup>1</sup> The predication *characterises*, like *injures* in the previous example, is expressed by a verb; but, as explained in Part I, Chapter XIII, section 5, any verb may be resolved into an adjective or relation preceded by the characterising tie. Thus, in order to show more explicitly that the principal constituents are united by the characterising tie, proposition (2) should be expanded into the form: ' $p$  is characterised as either characterising  $a$  or characterising  $b$  or characterising  $c$ .' Similarly for other examples.

(as in the preceding examples) into the form: ' $A$  has asserted  $p$  or has asserted  $q$  or has asserted  $r$ .'

(4) The proposition: ' $g$  is characterised by all the adjectives that characterise  $a$  and  $b$  and  $c$ ' exhibits a higher degree of complexity than those previously given since it introduces the two correlatives *characterising* and *characterised by*. It illustrates a type of proposition which plays an important part in the theory of induction; and is a specific case of the more general form: ' $g$  is  $\tilde{r}$  to everything that is  $\tilde{r}$  to  $a$  and  $b$  and  $c$ .' As thus formulated it contains the fictitious conjunctive construct ' $a$  and  $b$  and  $c$ ,' where  $a$ ,  $b$ ,  $c$  function as substantives. To eliminate this fictitious construct, the statement must be reformulated thus: ' $g$  is characterised by every adjective that characterises  $a$  and characterises  $b$  and characterises  $c$ .' But there still remains the fictitious construct prefaced by the distributive phrase 'every adjective.' The final correction must be made by introducing an apparent variable as was required in reformulating the elementary forms of proposition: 'Every  $M$  is  $p$ ,' 'Some  $M$  is  $p$ .' Thus: 'Every adjective, say  $x$ , that characterises  $a$  and characterises  $b$  and characterises  $c$  also characterises  $g$ .'

§ 12. The above exposition of functions is fundamentally opposed to that given in the *Principia Mathematica*. The first point of difference to be emphasised concerns Mr Russell's view of the relation between what he calls a propositional function, and function in the sense in which it is universally understood in mathematics. The latter he terms a descriptive function, and maintains that it is derivable from the nature of the propositional function; whereas it appears to me that

the reverse is the case, and that his propositional function is nothing but a particular case of the mathematical function. The general nature of a descriptive function can be illustrated by taking a proposition say about 'The teacher of  $y$ .' This phrase illustrates what is meant by a descriptive function, the full meaning of which can be indicated only by showing how it may enter into a proposition such as (a): 'The teacher of  $y$  was a Scotchman.' Now we may agree with Mr Russell that this proposition could not be interpreted as true, unless  $y$  had one and only one teacher. On this interpretation the full force of the proposition is explicated as follows:

(a) There is a being, say  $b$ , of which the following statements may be made:

- (1) that  $b$  was a Scotchman;
- (2) that  $b$  taught  $y$ ;
- (3) that no being other than  $b$  taught  $y$ .

This analysis in which the describing relation is *teaching* is typical of all cases in which a descriptive function is used in a proposition. To illustrate a mathematical function of  $y$ , for *teacher-of* substitute *greater-by-3-than*; so that  $y+3$  stands for 'the quantity that is greater by 3 than  $y$ .' Again for the predication *is-a-Scotchman* substitute *is-divisible-by-4*. Thus, in place of the proposition 'The teacher of  $y$  was a Scotchman,' we have constructed the proposition (b): ' $y+3$  is divisible by 4,' the full force of which is rendered as follows:

(b) There is a quantity, say  $b$ , of which the following statements may be made:

- (1) that  $b$  is divisible by 4;
- (2) that  $b$  is greater-by-3 than  $y$ ;
- (3) that no quantity other than  $b$  is greater-by-3 than  $y$ .

Thirdly, to illustrate a propositional function, for *divisible-by-4* substitute the predicate *dubious*; for the quantitative construct ' $y+3$ ' substitute the propositional construct ' $y$  is  $p$ .' We have thus constructed the secondary proposition ( $c$ ): 'That  $y$  is  $p$  is dubious,' of which the full force is rendered as follows:

( $c$ ) There is a proposition, say  $b$ , about  $y$ , of which the following statements may be made:

- (1) that  $b$  is dubious;
- (2) that  $b$  predicates- $p$ -about  $y$ ;
- (3) that no proposition other than  $b$  predicates- $p$ -about  $y$ .

Now in example ( $a$ ) the ground for asserting uniqueness of the construct *the teacher of  $y$*  is merely empirical or factual; but in example ( $b$ ) the necessary and sufficient condition for the uniqueness of the construct  $y+3$  is its mathematical form, as indicated by the symbol  $+$ ; and in example ( $c$ ) the uniqueness of the corresponding construct  $y$  is  $p$  similarly depends upon its *logical* form, as indicated by the logical constant *is*. Dismissing the empirical example which requires no further discussion, it must be pointed out as regards the quantitative function ( $b$ ) and the propositional function ( $c$ ) that these illustrate—not quantitative or propositional functions in general—but certain specific functions: in the former case that which is constructed by means of *plus*, and in the latter case, that which is constructed by means of *is*. The former may be called the *additive* and the latter the *characterising* function. Just as the quantitative construct  $y+a$  would not yield a quantity unless  $y$  and  $a$  were themselves quantities of the same kind; so the propositional construct  $y$  is  $p$  would not yield a proposition unless the two constituents  $y$  and  $p$  were, in

their nature, relatively to one another as substantive to adjective. A specifically different form of quantitative construct would have been obtained if for  $y+a$  we had substituted  $y:a$ . Similarly a specifically different propositional form of construct would have been obtained if for  $s$  is  $p$  we had substituted  $x$  is identical with  $y$ . In both cases the uniqueness of the construct is secured by the nature of the operator involved; viz.,  $+$  which yields a sum, or  $:$  which yields a ratio for the two quantitative constructs; and *is* and *is identical with* for the two propositional constructs. If there is any difference between the uniqueness of the propositional construct when its constituents are given and that of the mathematical construct when its constituents are given, it is that the uniqueness in the former case is assumed on the ground of its intuitive evidence realised in the mental act of constructing the proposition, whereas in the latter the uniqueness may require and may be capable of formal demonstration.

§ 13. Before continuing the discussion of my differences from Mr Russell, I shall examine more precisely what he means by a descriptive function. A descriptive function (p. 245) is defined to be a phrase of the form: '*the* term  $x$  that has the relation  $\hat{r}$  to the term  $y$ .' In this definition the sole emphasis is to be laid on the predesignation *the*. Now, just as we speak of *the* quantity ' $s+p$ ,' so we speak of *the* proposition ' $s$  is  $p$ .' But these quantitative and propositional phrases differ from ordinary descriptive phrases such as 'the writer of *Waverley*' or 'the teacher of Xenophon'—which are of the general form: 'the thing  $x$  that is  $\hat{r}$  to the thing  $y$ '—in that they do not explicitly contain any descriptive

relation  $\hat{r}$  (writing or teaching). The arithmetical form ' $s+p$ ' and the propositional form ' $s$  is  $p$ ' having in common this negative characteristic, I shall proceed to maintain that they are, in all essential logical respects, identical in nature; and, if either of the two can be explicated into the form of a descriptive function, so can the other. We may attempt to express these forms explicitly as descriptive functions by introducing, as the describing relation, *constructed by*. Thus the propositional function may be rendered: 'the proposition  $x$  constructed by means of *is* out of the constituents  $s$  and  $p$ '; and the quantitative function may be rendered: 'the quantity  $x$  constructed by means of *plus* out of the constituents  $s$  and  $p$ .' This attempt reduces the statement of equivalence of the construct with the proposed descriptive phrase to a mere tautology; for 'the proposition  $x$  constructed by means of *is* out of the constituents  $s$  and  $p$ ' is merely a lengthened expression for 'the proposition  $s$  is  $p$ '; and similarly 'the quantity  $x$  constructed by means of *plus* out of the constituents  $s$  and  $p$ ' is merely a lengthened expression for 'the quantity  $s+p$ .' It thus turns out that the  $x$  thus introduced in the completely formulated descriptive phrase stands merely for the function itself, i.e. in the one case for ' $s$  is  $p$ ' and in the other for ' $s+p$ .' Following Russell in his demand that a descriptive function must only be defined 'in use,' the statement that ' $s$  is  $p$  is dubious' or that ' $s+p$  is divisible by 4' must be rendered 'the proposition  $x$  constructed etc. (as above) is dubious,' or 'the quantity  $x$  constructed etc. (as above) is divisible by 4.' In this way the original statements: 'the proposition  $s$  is  $p$  is dubious,' and 'the quantity

$s+p$  is divisible by 4,' which were supposed to require definition, are after all defined tautologically.

§ 14. Another way of attacking Russell's propositional function, which in fact presents only another aspect of the same criticism, is to ask: What are the variants for any given propositional function, and what function is it that a given propositional form exhibits? In his first introduction of the notion of propositional function, Mr Russell gives three quite different applications of the symbol for a function. According to his first definition,  $\phi x$  is called a propositional function when  $x$  is variable provided that when  $x$  is replaced by the constant  $a$ ,  $\phi a$  represents a proposition. Now here the symbol for a function is first used along with a variable and then along with a constant; although Russell insists that  $\phi a$  is not a function but a proposition, and that  $\phi x$  is not a proposition but a function. It seems to me that he cannot attach the symbol for a function exclusively to a variable in this way without contradiction at every point; and it is for this reason that, in my account of functions, I have used the word *variant* to include both Russell's variable and his constant. There is yet a third application of the symbol for a function deliberately introduced in the very first paragraph of his exposition, by way of correcting his initial definition of propositional function. For his first account is that  $\phi x$  is to be called a propositional function, owing to the ambiguity—or as I should prefer to say indeterminateness—of the symbol  $x$ , and that it is not itself a proposition, and would only become a proposition when  $a$  is substituted for  $x$ . This is corrected, however, when he takes the example ' $x$  is hurt' which he says illustrates, not a propositional

function, but an ambiguous (i.e. an indeterminate) value of a propositional function. Thus, as I have pointed out, he illustrates the use of the word *function* in his first paragraph in three different ways which are symbolised as follows: '*a* is hurt,' '*x* is hurt,' and '*x̂* is hurt.' The last application of the word function is that which he wishes to be finally adopted; but, in spite of this, he continually uses the word in both of the two other applications. It is still more surprising that, on page 6 of his Introduction, where he gives a preliminary account of the ideas and notations of logical symbolism, he uses the word function without any explanation of its meaning, and in deliberate defiance of his own later definition. Thus he speaks of the fundamental functions of propositions in these words: 'an aggregation of propositions considered as wholes, not necessarily unambiguously determined, into a single proposition more complex than its constituents, is a function with propositions as arguments.' This account appears clearly to suggest that unambiguously determined constituents are allowable as arguments for a function, which contradicts his explicit definition. He proceeds to enumerate the four fundamental functions of propositions which are of logical importance, viz. (1) the contradictory function, which I have called the negative function; (2) the logical sum or disjunctive function, which I have called the alternative function; (3) the logical product, which both he and I call the conjunctive function; (4) the implicative function, for which I have used the same term. These four functions I have called conjunctive functions, in contrast to the one fundamental predication function. The recognition of this distinction, which does not appear

in Mr Russell's account, would have simplified and corrected his 'theory of types.' But, in thus introducing the specific conjunctive functions, he inevitably adopts the familiar meaning of the mathematical term 'function,' the essence of which lies, not in the indeterminateness of the constituent terms, but in the identity of form that is exhibited in the process of substituting indifferently any one term for any other.

That he is not only in disagreement with universal usage, but also logically mistaken, when he says that it is a *function* of which the essential characteristic is ambiguity—and thus that  $\phi x$  ambiguously denotes  $\phi a$ ,  $\phi b$ ,  $\phi c$ , where  $\phi a$ ,  $\phi b$ ,  $\phi c$  are the various values of  $\phi x$ —is shown by noting that the ambiguity attaching to  $\phi x$  is not due to the nature of  $\phi$  as a function, but to the nature of the symbol  $x$  itself; that is to say,  $\phi x$  ambiguously denotes  $\phi a$ ,  $\phi b$ ,  $\phi c$ , etc., only because  $x$  ambiguously denotes  $a$ ,  $b$ ,  $c$ , etc. In short a propositional function has ambiguous denotation, if it contains a term having ambiguous denotation; whereas a propositional function has unambiguous denotation, if it contains no term having ambiguous denotation.

§ 15. Hitherto, in illustrating Russell's account, we have taken the propositional function to be a function of a single variable, viz., of the symbol for the subject of the proposition, the predicate standing for a constant. It is obvious, however, that no proposition can be regarded as a function of a single variant unless the proposition is represented by a simple letter; and we will therefore take the specific propositional form '*x* is *p*' to illustrate a function of two variables. The variants of which this is a function would naturally be taken as the



symbols  $x$  and  $p$  themselves; but, since Russell refuses to allow a predicate or adjective to stand by itself, he takes as the two variables the subject term  $x$  together with the symbolic variable ' $\hat{x}$  is  $p$ .' The symbolic expression ' $\hat{x}$  is  $p$ ' may be read ' $x$ -blank is  $p$ '; by which is meant that instead of the full propositional form ' $x$  is  $p$ ,' we suppose that the subject-term  $x$  is omitted, leaving a blank. But, if we use a blank symbol for the subject-term, we ought in consistency to be allowed to use a similar blank symbol for the predicate term. This would give rise to nine combinations all of which are of the same propositional form: 'this is hurt,' ' $x$  is hurt,' ' $x$  is  $p$ ,' ' $\hat{x}$  is hurt,' ' $\hat{x}$  is  $p$ ,' ' $x$  is  $\hat{p}$ ,' ' $\hat{x}$  is  $\hat{p}$ ,' and finally ' $x$  is  $p$ .' Of these nine phrases, Russell uses only 'this is hurt,' ' $x$  is hurt' and ' $\hat{x}$  is hurt'; of which the two latter illustrate the two admittedly different meanings or applications of the general notion  $\phi x$ , i.e. of the propositional function. Now, though his first reference is to a propositional function taking a single argument, nevertheless he allows that any proposition (as distinguished from a propositional function) when analysed contains at least two constituents. For example, the proposition 'this is hurt' as analysed contains the two constituents 'this' and ' $\hat{x}$  is hurt.' In my view, there is no ground whatever for preferring this analysis either to that in which the constituents are 'hurt' and ' $\hat{x}$  is  $p$ ,' or to that in which the constituents are 'this is  $p$ ' and ' $\hat{x}$  is hurt.' But, returning to his own analysis in which 'this' and ' $\hat{x}$  is hurt' are assigned as the two constituents of 'this is hurt,' as also ' $x$ ' and ' $\hat{x}$  is  $p$ ' as the two constituents of ' $x$  is  $p$ '; we must insist upon asking: What is the specific function for the

case of the proposition ' $x$  is  $p$ ' when its two arguments are taken to be ' $x$ ' and ' $\hat{x}$  is  $p$ '? Mr Russell only tells us that ' $x$  is  $p$ ' = ' $f(x, \hat{x} \text{ is } p)$ ' where the specific symbol  $f$  has nowhere been defined by him. It is as if he had said that the quantitative function ' $x + p$ ' has for its two constituents, variants or arguments: (1)  $x$  and (2)  $\hat{x} + p$ . Now according to this analysis of the nature of a function, the process by which a function is constructed out of two variables is to substitute in one of these variables  $x$  for  $\hat{x}$ , so that taking a similar example to the above, the constituents of the quantitative construct ' $x \div p$ ' would be  $x$  and  $\hat{x} \div p$ . Every mathematician would take as the two constituents of the construct  $x \div p$  the two simple symbols  $x$  and  $p$ ; as Russell himself does in his preliminary account of the alternative function  $x$  or  $p$ , of which the two constituents are the simple symbols  $x$  and  $p$ . In fact he can only take a function of a single variable as ambiguously denoting a proposition, by starting with what I have called a non-formal function, e.g. ' $x$  is hurt' as a non-formal function of  $x$ ; instead of starting with the essentially logical notion of a function, which is synonymous with the form of a construct such as ' $x$  is  $p$ ' where instead of one material or variable constituent there are two. In short the form of a proposition, if it has form at all and is not simply expressed by a simple symbol, must contain two independent constituents. When Mr Russell says that  $\phi(x)$  is a propositional function, provided that  $\phi(a)$  is a proposition, he provides us with no indication as to the *form* that  $\phi(a)$  must assume in order that  $\phi(a)$  shall constitute a proposition.



## CHAPTER IV

### THE CATEGORICAL SYLLOGISM

§ 1. As the relation between implication and inference has already been explained, we may treat the syllogism indifferently as a species either of implication or of inference: regarded as implication, the propositions concerned must be spoken of as implicants and implicate; regarded as inference, we speak of them as premisses and conclusion. The term syllogism is strictly confined to one only of the many forms of demonstrative inference; and in this strict usage must be defined as an argument containing two premisses and a conclusion, involving between them three terms, each of which occurs in two different propositions. That occurring as predicate in the conclusion is called the major term; that occurring as subject in the conclusion, the minor term; and that not occurring in the conclusion, the middle term. The distinction between the major and minor terms determines which of the premisses shall be called major and which minor: that which contains the predicate of the conclusion being called the major premiss; and that which contains the subject of the conclusion being called the minor premiss. Reference to the conclusion is thus required before the premisses can be distinguished as major or minor. The canonical order of the three propositions, viz. major premiss, minor premiss, conclusion, is purely artificial, and adopted only for general purposes of reference. The mood of a syllogism is defined by the forms (*A*, *E*, *I*,

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or *O*) of the three propositions constituting the major premiss, minor premiss, and conclusion, in their canonical order. Furthermore syllogisms are distinguished according to figure: the first figure being that in which the middle term occurs as subject in the major premiss and predicate in the minor; the fourth figure being that in which the middle term occurs as predicate in the major and as subject in the minor: the second figure, that in which the middle term occurs as predicate in both premisses; and the third figure that in which the middle term occurs as subject in both premisses. Two syllogisms would be said to be of different form, although they might agree in mood, provided they differed in figure.

§ 2. There are two opposite tendencies in the choice of illustrations of the syllogism, both of which, in my view, should be avoided. The first is to select examples composed of propositions, each of which is universally accepted as true. But such illustrations hinder the learner from examining the validity of the inferential process from premisses to conclusion, since he is apt to assume validity because of his familiarity with the propositions as being generally accepted. The opposite course, which we find amusingly illustrated by Lewis Carroll, is to select propositions which are obviously false. But this leads the learner to regard the syllogism merely as a kind of game, and as having no real significance in actual thought procedure. It is preferable, therefore, to select propositions which are dubious, or which are affirmed by some persons and denied by others. Of such propositions important kinds are (1) those which deal with political, ethical, or similar

topics in *general*, e.g. 'Lying is sometimes right,' 'All countries that adopt free-trade are prosperous,' 'The suffrage should not be extended to uneducated persons; (2) those which exercise the faculty of judgment, in the Kantian sense, upon some *individual* case, e.g. 'This man is untrustworthy,' 'The Niche is finer than the Venus of Milo,' 'Esau is a more lovable person than Jacob.'

§ 3. Correlative to the syllogism we may here introduce the *antilogism*, in reference to which the above principle of selecting examples will be seen to have special significance. An antilogism may be defined as a formal disjunction of two, three, or more propositions, each of which is entertained hypothetically. When limited to three propositions constituting a disjunctive trio, the antilogism may be formulated in terms of illustrative symbols as follows: 'the three propositions *P*, *Q*, and *R* cannot be true together.' It is then seen that just as the disjunction of *P* and *Q* is equivalent to the implication 'If *P* is true, then *Q* is false,' so the disjunction of *P*, *Q*, and *R* is equivalent to each of the three implications:

- (1) If *P* and *Q* are true, then *R* is false,
- (2) If *P* and *R* are true, then *Q* is false,
- (3) If *R* and *Q* are true, then *P* is false.

We may put forward the following example of an antilogism, no one of the propositions of which would be universally acknowledged either as true or as false, but which taken together are formally incompatible:

- P*. All tactful persons sometimes lie.
- Q*. Lord Grey is a tactful person.
- R*. Lord Grey never lies.

Something could be said in support of, as well as in opposition to, each of these three propositions; but it is obvious that they are together incompatible, and hence constitute an antilogism or disjunctive trio. This antilogism is equivalent to each of the three following syllogistic implications:

- 1st if All tactful persons sometimes lie  
and Lord Grey is a tactful person,  
then Lord Grey sometimes lies.
- 2nd if All tactful persons sometimes lie  
and Lord Grey never lies,  
then Lord Grey is not a tactful person.
- 3rd if Lord Grey never lies  
and Lord Grey is a tactful person,  
then Some tactful persons never lie.

§ 4. The propositions in each of these syllogisms are in the canonical order of major, minor, conclusion, and the syllogisms will be recognised as being in the first, second, and third figures respectively. In defining the figures of syllogism we may, in fact, separate the first three from the fourth in that the former contain one and only one term standing in one proposition as subject and in another as predicate, while in the fourth figure all three terms occupy this double position. Such a term may be called a class-term, on the ground that a class-term has a partly adjectival meaning, and as such serves appropriately as predicate; and partly a substantival meaning, and as such serves appropriately as subject. The first three figures, then, containing only one class-term, are distinguished from one another according as this term occupies one or another position. In the first figure it serves as the middle term; in the

second figure as the major term; and in the third figure as the minor term. Taking the above antilogism as illustrative, we may generalise by formulating the following antilogistic dictum for the first three figures:

It is impossible to conjoin together the three propositions:

Every member of a class has a certain property;  
A certain object is included in that class;  
This object has not that property.

From this single antilogistic dictum we construct the dicta for the first three figures of syllogism, thus:

*Dictum for 1st Figure*

if Every member of a class has a certain property  
and A certain object is included in that class,  
then This object must have that property.

*Dictum for 2nd Figure*

if Every member of a class has a certain property  
and A certain object has not that property,  
then This object must be excluded from the class.

*Dictum for 3rd Figure*

if A certain object has not a certain property  
and This object is included in a certain class,  
then Not every member of the class has that property.

These dicta bring out the normal function of each of the first three figures in thought-process. Thus we are reasoning in the first figure when, having established a certain characteristic as belonging to every member of a class, we bring forward an individual object known to belong to the class and proceed to assert that it will have the characteristic common to the class. We are reasoning in the second figure when, having similarly

established a certain characteristic as belonging to every member of a class, and having found that an individual object has not this characteristic, we proceed to assert that it does not belong to the class. We are reasoning in the third figure when we note that a certain object known to belong to a certain class has not a certain property, and proceed to assert that that property cannot be predicated universally of all members of the class; or otherwise, when, having noted that an object known to belong to a certain class has a certain character, we infer that at least one member of the class has this character. A peculiarity of the third figure is that it functions either destructively or constructively; as destructive, it disproves some universal proposition that may have been suggested; as constructive, it naturally suggests the replacement of the particular conclusion either by a universal whose subject is restricted by some further adjectival characteristic, or by an unrestricted universal to be obtained by induction *from* the particular conclusion.

§ 5. A second illustration of an antilogism developing into three syllogisms may be chosen with the purpose of showing how purely formal and elementary reasoning underlies even the most abstract arguments. Thus:

It is impossible to conjoin the three propositions:

- P.* All possible objects of thought are such as have been sensorially impressed upon us;
- Q.* Substance is a possible object of thought;
- R.* Substance has not been sensorially impressed upon us.

Since each of these propositions has been asserted by

some and denied by other philosophers, the three together constitute an antilogism having the same illustrative value as our previous example.

Taking, first, *P* and *Q* as asserted premisses and not-*R* as conclusion, we obtain the syllogistic inference:

*P*. All possible objects of thought have been sensorially impressed upon us;

*Q*. Substance is a possible object of thought;

∴ not-*R*. Substance has been sensorially impressed upon us.

With some explanations and modifications this syllogism represents roughly one aspect of the new realistic philosophy.

Taking, next, *P* and *R* as asserted premisses and not-*Q* as conclusion, we have:

*P*. All possible objects of thought have been sensorially impressed upon us;

*R*. Substance has not been sensorially impressed upon us;

∴ not-*Q*. Substance is not a possible object of thought. This syllogism represents very fairly the position of Hume.

Taking, lastly, *R* and *Q* as asserted premisses and not-*P* as conclusion, we have:

*R*. Substance has not been sensorially impressed upon us;

*Q*. Substance is a possible object of thought;

∴ not-*P*. Not every possible object of thought has been sensorially impressed upon us.

This syllogism represents almost precisely the well-known position of Kant.

As in our previous example these three syllogisms are respectively in figures 1, 2, and 3; and, moreover, Kant's argument in figure 3 has both a destructive function in upsetting Hume's position; and a constructive function in suggesting the replacement of the particular conclusion by a limited universal which would assign the further characteristic required for discriminating those objects of thought which have not been obtained by experience from those which have been thus obtained.

§ 6. Since the dicta, as formulated above, apply only where two of the propositions are singular or instancial, they must be reformulated so as to apply also where *all* the propositions are general, i.e. universal or particular. Furthermore, they will be adapted so as to determine directly all the possible variations for each figure. As follows:

*Dictum for Fig. 1*

if *Every* one of a certain class *C* possesses (or lacks) a certain property *P*  
and Certain objects *S* are *included* in that class *C*,  
then These objects *S* must possess (or lack) that property *P*.

*Dictum for Fig. 2*

if *Every* one of a certain class *C* possesses (or lacks) a certain property *P*  
and Certain objects *S* lack (or possess) that property *P*,  
then These objects *S* must be *excluded* from the class *C*.

*Dictum for Fig. 3*

if Certain objects *S* possess (or lack) a certain property *P*

and These objects *S* are *included* in a certain class *C* then *Not every* one of the class *C* lacks (or possesses) that property *P*.

i.e. *Some* of the class *C* possess (or lack) that property *P*.

In each of these dicta the word 'objects,' symbolised as *S*, represents the term that stands as subject in both its occurrences; the word 'property' *P*, the term that stands as predicate in both its occurrences; and the word 'class' *C*, that term which occurs once as subject and again as predicate. Hence, using the symbols *S*, *C*, *P*, the first three figures are thus schematised:

|                     |                     |                     |
|---------------------|---------------------|---------------------|
| Fig. 1              | Fig. 2              | Fig. 3              |
| $C - P$             | $C - P$             | $S - P$             |
| $\underline{S - C}$ | $\underline{S - P}$ | $\underline{S - C}$ |
| $\therefore S - P$  | $\therefore S - C$  | $\therefore C - P$  |

§ 7. In order systematically to establish the moods which are valid in accordance with the above dicta, it should be noted in each figure (1) that the proposition *S - P* is unrestricted as regards both quality and quantity; (2) that the proposition *S - C* is independently fixed in quality, but determined in quantity by the *quantity* of the unrestricted proposition; and (3) that the proposition *C - P* is independently fixed in quantity, but determined in quality by the *quality* of the unrestricted proposition. Thus in Fig. 1, while the conclusion is unrestricted, the minor premiss is independently fixed in quality but determined in quantity by the quantity of the conclusion; and the major premiss is independently fixed in quantity but determined in quality by the quality of the conclusion. In Fig. 2, while the minor premiss is unrestricted, the conclusion

is independently fixed in quality but determined in quantity by the quantity of the minor premiss; and the major premiss is independently fixed in quantity, but determined in quality by the quality of the minor premiss. In Fig. 3, while the major premiss is unrestricted, the minor premiss is independently fixed in quality but determined in quantity by the quantity of the major premiss, and the conclusion is independently fixed in quantity but determined in quality by the quality of the major premiss.

Having in the above dicta italicised the phrase in each case which is directly restrictive, the proposition which is unrestricted, i.e. may be of the form *A* or *E* or *I* or *O*, is seen to be: in Fig. 1, the conclusion; in Fig. 2, the minor premiss; in Fig. 3, the major premiss. Hence each of these figures contains four fundamental moods derived respectively by giving to the unrestricted proposition the form *A*, *E*, *I* or *O*. Besides these four fundamental moods there are also supernumerary moods. These are obtained by substituting, in the conclusion, a particular for a universal; or, in the minor premiss, a universal for a particular; or, in the major again, a universal for a particular. These supernumerary moods will be said respectively to contain a weakened conclusion, a strengthened minor, or a strengthened major; and, in the scheme given in the next section, the propositions thus weakened or strengthened will be indicated by the raised letters *w* or *s* as the case may be.

§ 8. Adopting the method above explained, we may now formulate the special rules for determining the valid moods in each figure as follows:



*Rules for Fig. 1.*

The conclusion being unrestricted in regard both to quality and quantity,

- (a) The major premiss must in quantity be universal, and in quality agree with the conclusion.
- (b) The minor premiss must be in quality affirmative, and in quantity as wide as the conclusion.

*Rules for Fig. 2.*

The minor premiss being unrestricted in regard both to quality and quantity,

- (a) The major premiss must be in quantity universal, and in quality opposed to the minor.
- (b) The conclusion must be in quality negative, and in quantity as narrow as the minor.

*Rules for Fig. 3.*

The major premiss being unrestricted in regard both to quantity and quality,

- (a) The conclusion must in quantity be particular, and in quality agree with the major,
- (b) The minor premiss must in quality be affirmative, and in quantity overlap<sup>1</sup> the major.

Italicising in each case the unrestricted proposition, we may represent the valid moods for the first three figures in the following table:

*Valid Moods for the "One-Class" Figures.*

|        | Fundamentals |     |     |     | Supernumeraries   |                   |
|--------|--------------|-----|-----|-----|-------------------|-------------------|
| Fig. 1 | AAA          | EAE | AII | EIO | AAI <sup>SW</sup> | EAO <sup>SW</sup> |
| Fig. 2 | EAE          | AEE | EIO | AOO | EAO <sup>SW</sup> | AEO <sup>SW</sup> |
| Fig. 3 | AII          | EIO | IAI | OOO | AAI <sup>SS</sup> | EAO <sup>SS</sup> |

<sup>1</sup> The minor and major will necessarily overlap if one or the other is universal, not otherwise.

§ 9. Having established the valid moods of the first three figures from a single antilogism, we proceed to construct those of the fourth figure also from a single antilogism; thus:

Taking any three classes, it is impossible that

The first should be wholly included in the second while The second is wholly excluded from the third and The third is partly included in the first.

The validity of this antilogism is most naturally realised by representing classes as closed figures. Such a representation is in fact valid, although the relation of inclusion and exclusion of classes is not identical with the logical relations expressed in affirmative and negative propositions respectively; for, there is a true analogy between the relations between classes and the relations between closed figures; in that the relations between the relations of classes are identical with the corresponding relations between the relations of closed figures. Thus adopting as the scheme of the fourth figure:

$$C_1 - C_2 \quad C_2 - C_3 \quad C_3 - C_1$$

the above antilogism will be thus symbolised:

It is impossible to conjoin the following three propositions:

*P.* Every  $C_1$  is  $C_2$ ,

*Q.* No  $C_2$  is  $C_3$ ,

*R.* Some  $C_3$  is  $C_1$ .

This yields the three fundamental syllogisms

(1) If *P* and *Q*, then not-*R*; i.e.

if Every  $C_1$  is  $C_2$

and No  $C_2$  is  $C_3$ ,

then No  $C_3$  is  $C_1$ .

B D

A C

C B



- (2) If  $Q$  and  $R$ , then not- $P$ ; i.e.  
     if No  $C_2$  is  $C_3$   
     and Some  $C_3$  is  $C_1$ ,  
     then Not every  $C_1$  is  $C_2$ .
- (3) If  $R$  and  $P$ , then not- $Q$ ; i.e.  
     if Some  $C_3$  is  $C_1$   
     and Every  $C_1$  is  $C_2$ ,  
     then Some  $C_2$  is  $C_3$ .

Since the propositions of these syllogisms are arranged in canonical order, the valid moods in the fourth figure can be at once written down: *AEE, EIO, IAI*. Moreover, since the conclusion of the first mood is universal, it may be weakened; since the minor of the second is particular, it may be strengthened; and since the major of the third is particular, it also may be strengthened. This yields:

*Valid Moods of the Fourth Figure.*

| Fundamentals |     |     | Supernumeraries  |                  |                  |
|--------------|-----|-----|------------------|------------------|------------------|
| AEE          | EIO | IAI | <sup>W</sup> AEO | <sup>S</sup> EAO | <sup>S</sup> AAI |

Here each supernumerary can only be interpreted in one sense, viz., as containing respectively a weakened conclusion, a strengthened minor, and a strengthened major. In contrast to this, the supernumeraries of the first and second figures must be interpreted as containing *either* a weakened conclusion *or* a strengthened minor; and those of the third figure as containing *either* a strengthened major *or* a strengthened minor.

§ 10. An antiquated prejudice has long existed against the inclusion of the fourth figure in logical doctrine, and in support of this view the ground that has been most frequently urged is as follows:

Any argument worthy of logical recognition must be such as would occur in ordinary discourse. Now it will be found that no argument occurring in ordinary discourse is in the fourth figure. Hence, no argument in the fourth figure is worthy of logical recognition.

This argument, being in the fourth figure, refutes itself; and therefore needs to be no further discussed.

§ 11. Having formulated certain intuitively evident dicta, the observance of which secures the validity of the syllogisms established by their means, we will proceed to formulate equally intuitive rules the violation of which will render syllogisms invalid. These rules will be found to rest upon a single fundamental consideration, viz. if our data or premisses refer to *some* only of a class, no conclusion can be validly drawn which refers to *all* members of that class. This is technically expressed in the rule:

(1) 'No term which is undistributed in its premiss may be distributed in the conclusion.'

This rule *alone* is not sufficient *directly* to secure validity, but from it we can deduce other directly applicable rules which, taken in conjunction with the first, will be sufficient to establish directly the invalidity of any invalid form of syllogism. In the course of deducing these other rules we shall make use of certain logical intuitions that are obvious apart from their employment in this deductive process, of which the following may be mentioned:

(a) that if a term is distributed in any given proposition, it will be undistributed in the contradictory proposition; and conversely, if a term is undistributed in a given proposition, it will be distributed in the

contradictory proposition. That this is so is directly seen when it has been accepted on intuitive grounds that only universals distribute the subject term, and only negatives the predicate term; and that an *A* proposition is contradicted by an *O*, and an *I* proposition by an *E*.

(*b*) That any syllogism can be expressed as an antilogism and conversely. This principle follows from the intuitive apprehension of the relation between implication and disjunction.

(*c*) That it is formally possible for any three terms to coincide in extension. (This particular intuition is employed in the rejection of only one form of syllogism.)

We are now in a position to deduce from our original principle, i.e. from rule (1), by means of (*a*), (*b*), and (*c*), other rules, the direct application of which will exclude any invalid forms of syllogism.

(2) 'The middle term must be distributed in one or other of the premisses.'

To establish this, let us consider the antilogism which disjoins *P*, *Q* and *R*; this, by (*b*) is equivalent to the syllogism 'If *P* and *Q*, then not-*R*' and also to the syllogism 'If *P* and *R*, then not-*Q*.' Taking the first of these, if a term *X* is undistributed in the premiss *P*, it must be undistributed in the conclusion not-*R*, i.e. it must, by (*a*), be distributed in *R*. Applying this result to the second syllogism 'If *P* and *R*, then not-*Q*,' we have shown that if the middle term *X* is undistributed in the premiss *P*, it must be distributed in the premiss *R*. This then establishes rule (2).

(3) 'If both premisses are negative, no conclusion can be syllogistically inferred.'

For, taking any two universal negative premisses, these can be converted (if necessary) into 'No *P* is *M*' and 'No *S* is *M*'; which, by obversion, are respectively equivalent to 'All *P* is non-*M*' and 'All *S* is non-*M*,' in which the new middle term non-*M* is undistributed in both premisses. But this breaks rule (2). What holds of two universals will hold *a fortiori* if one or other of the two negative premisses is particular. Thus rule (3) is established.

(4) 'A negative premiss requires a negative conclusion.'

For, taking again the antilogism which disjoins *P*, *Q* and *R*, this is equivalent both to the syllogism 'If *P* and *R*, then not-*Q*,' and to the syllogism 'If *P* and *Q*, then not-*R*.' Taking the first of these two syllogisms, by rule (3), if the premiss *P* is negative, the premiss *R* must be affirmative. Applying this result to the second syllogism, we have, if the premiss *P* is negative, the conclusion not-*R* must be negative. This establishes rule (4).

(5) 'A negative conclusion requires a negative premiss.'

This is equivalent to the statement that two affirmative premisses cannot yield a negative conclusion. To establish this rule, we must take the several different figures of syllogism:

| Fig. 1       | Fig. 2       | Fig. 3       | Fig. 4       |
|--------------|--------------|--------------|--------------|
| M - P        | P - M        | M - P        | P - M        |
| <u>S - M</u> | <u>S - M</u> | <u>M - S</u> | <u>M - S</u> |
| S - P        | S - P        | S - P        | S - P        |

For the first or third figure, affirmative premisses with negative conclusion would entail false distribution

of the major term; which has been forbidden under our fundamental rule (1). Taking next the second figure, it would entail false distribution of the middle term, forbidden by rule (2). Finally taking the fourth figure, it would either entail some false distribution forbidden by rules (1) and (2); or else yield the mood *AAO* which would constitute a denial that three terms could coincide in extension, thus contravening (c). This establishes rule (5).

§ 12. The five rules thus established may be re-arranged and summed up into two rules of quality and two rules of distribution, viz.

A. *Rules of Quality.*

- (a<sub>1</sub>) For an affirmative conclusion both premisses must be affirmative.
- (a<sub>2</sub>) For a negative conclusion the two premisses must be opposed in quality.

B. *Rules of Distribution.*

- (b<sub>1</sub>) The middle term must be distributed in at least one of the premisses.
- (b<sub>2</sub>) No term undistributed in its premiss may be distributed in the conclusion.

These rules having been framed with the purpose of rejecting invalid syllogisms, we may first point out that, irrespective of validity, there are sixty-four abstractly possible combinations of major, minor and conclusion. The Rules of Quality enable us to reject *en bloc* all moods except those coming under the following three heads, viz. those which contain (i) an *affirmative conclusion* (requiring affirmative major and affirmative minor); (ii) a *negative major* (requiring affirmative

minor and negative conclusion); (iii) a *negative minor* (requiring affirmative major and negative conclusion). This leads to the following table, which exhibits the 24 possibly valid moods unrejected by the Rules of Quality.

|             | Maj. & Min.<br>Univ. |     | Maj.<br>Univ. | Min.<br>Part. | Maj.<br>Part. | Min.<br>Univ. | Maj. & Min.<br>Part. |     |
|-------------|----------------------|-----|---------------|---------------|---------------|---------------|----------------------|-----|
| Concl. Aff. | AAI                  | AAA | AII           | AIA           | IAI           | IAA           | III                  | IIA |
| Maj. Neg.   | EAO                  | EAE | EIO           | EIE           | OAO           | OAE           | OIO                  | OIE |
| Min. Neg.   | AEO                  | AEE | AOO           | AOE           | IEO           | IEE           | IOO                  | IOE |

§ 13. The Rules of Quality having thus been applied, it remains to reject such of the above 24 as violate the Rules of Distribution, (i) for the middle term, (ii) for the major term, (iii) for the minor term. This will require three special rules for each of the four figures:

|              |              |              |              |
|--------------|--------------|--------------|--------------|
| Fig. 1       | Fig. 2       | Fig. 3       | Fig. 4       |
| M - P        | P - M        | M - P        | P - M        |
| <u>S - M</u> | <u>S - M</u> | <u>M - S</u> | <u>M - S</u> |
| S - P        | S - P        | S - P        | S - P        |

The above scheme shows that it will be convenient to bracket Fig. 1 with Fig. 4 for the *middle term*, Fig. 1 with Fig. 3 for the *major term*, and Fig. 1 with Fig. 2 for the *minor term*; leading to the following:

RULES FOR CORRECT DISTRIBUTION.

1st of the *Middle Term*.

- Fig. 1. If the minor is affirmative, the major must be universal.
- Fig. 4. If the major is affirmative, the minor must be universal.

- Fig. 2. One premiss must be negative; i.e. *conclusion must be negative*.  
 Fig. 3. One or the other of the premisses must be universal.

*2nd of the Major Term.*

Figs. 1 and 3. If the conclusion is negative, the major must be negative; i.e. (in either case) *the minor must be affirmative*.

Figs. 2 and 4. If the conclusion is negative, the major must be universal.

*3rd of the Minor Term.*

Figs. 1 and 2. If the minor is particular, the conclusion must be particular.

Figs. 3 and 4. If the minor is affirmative, the conclusion must be particular.

These rules have been grouped by reference to the *term* (middle, major or minor) which has to be correctly distributed. They will now be grouped by reference to the *figure* (1st, 2nd, 3rd or 4th) to which each applies. In this rearrangement we shall also simplify the formulations by replacing where possible a hypothetically formulated rule by one categorically formulated. As a basis of this reformulation we take the rules of quality for Figs. 1, 2 and 3, which have already been expressed categorically; viz. for Figs. 1 and 3: 'The minor premiss must be affirmative,' and for Fig. 2: 'The conclusion must be negative.' Conjoining the categorical rule (of quality) for Fig. 1 with its hypothetical rule, 'If the minor is affirmative the major must be universal,' we deduce for this figure the categorical rule (of quantity), 'The major must be universal.' Again, conjoining the

The Special Rules of Distribution for each Figure are:

| Fig. 1                                              |                                                     |                                   | Fig. 2                                              |                                   |                                                      | Fig. 3 |  |  | Fig. 4 |  |  |
|-----------------------------------------------------|-----------------------------------------------------|-----------------------------------|-----------------------------------------------------|-----------------------------------|------------------------------------------------------|--------|--|--|--------|--|--|
| a. Minor affirmative.                               | a. Conclusion negative.                             | a. Minor affirmative.             | a. Conclusion negative.                             | a. Minor affirmative.             | 1. If major affirmative, then minor universal.       |        |  |  |        |  |  |
| b. Major universal.                                 | b. Major universal.                                 | b. Conclusion particular.         | b. Major universal.                                 | b. Conclusion particular.         | 2. If conclusion negative, then major universal.     |        |  |  |        |  |  |
| c. If minor particular, then conclusion particular. | c. If minor particular, then conclusion particular. | c. One premiss must be universal. | c. If minor particular, then conclusion particular. | c. One premiss must be universal. | 3. If minor affirmative, then conclusion particular. |        |  |  |        |  |  |

Applying these rules of distribution to the scheme of possibly valid moods unrejected by the rules of quality, we have :

Fig. 1. Rule *a* rejects last row (8 moods)

Rule *b* rejects last 4 columns (12 moods)

Rule *c* rejects 4th and 8th columns (6 moods)

Fig. 2. Rule *a* rejects 1st row (8 moods)

Rule *b* rejects last 4 columns (12 moods)

Rule *c* rejects 4th and 8th columns (6 moods)

Fig. 3. Rule *a* rejects last row (8 moods)

Rule *b* rejects even-numbered columns (12 moods)

Rule *c* rejects last two columns (6 moods)

Fig. 4. Rule 1 rejects from 1st and 3rd row the 3rd, 4th, 7th and 8th columns (8 moods)

Rule 2 rejects from 2nd and 3rd row the 5th, 6th, 7th and 8th columns (8 moods)

Rule 3 rejects from 1st and 2nd row the 2nd, 4th, 6th and 8th columns (8 moods)

| Unrejected by distribution |     |               |
|----------------------------|-----|---------------|
| ...                        | ... | { AAI AAA AII |
| ...                        | ... | { EAO EAE EIO |
| ...                        | ... | ...           |
| ...                        | ... | { EAO EAE EIO |
| ...                        | ... | { AEO AEE AOO |
| ...                        | ... | { AAI AII IAI |
| ...                        | ... | { EAO EIO OAO |
| ...                        | ... | ...           |
| ...                        | ... | ...           |
| ...                        | ... | AAI           |
| ...                        | ... | EAO           |
| ...                        | ... | AEO AEE       |

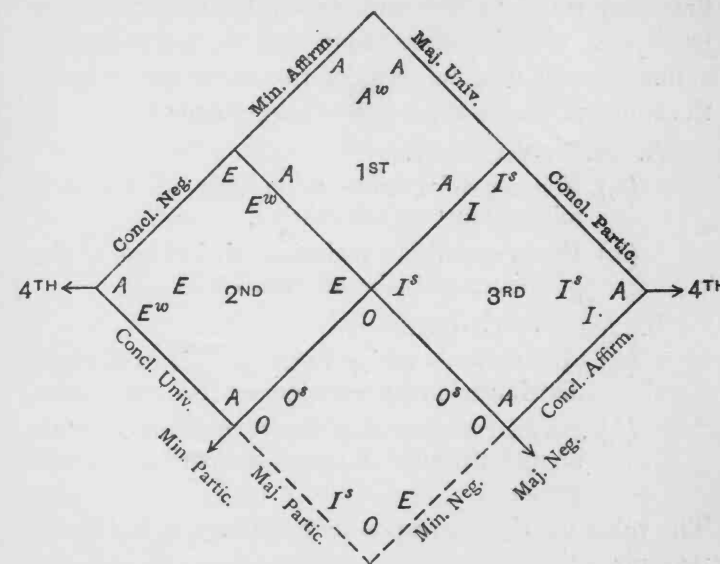
categorical rule (of quality) for Fig. 2 with its hypothetical rule 'If the conclusion is negative the major must be universal,' we deduce for this figure the categorical rule (of quantity), 'The major must be universal.' Lastly, conjoining the categorical rule (of quality) for Fig. 3 with its hypothetical rule, 'If the minor is affirmative the conclusion must be particular,' we deduce the categorical rule (of quantity) for this figure, 'The conclusion must be particular.' The remaining rules must be repeated without modification.

The *Special Rules of Distribution for each Figure* and the application of these rules of distribution to the scheme of possibly valid moods unrejected by the rules of quality are set out on the preceding page.

§ 14. We will now compare the results reached by the two methods—direct and indirect. The direct method determines, by means of certain intuited dicta, what moods are to be accepted as valid; the indirect method determines—on equally intuitive principles—what moods are to be rejected as invalid, and consequently what moods remain unrejected. We gather from this comparison that the 24 moods (6 for each figure) that are established as valid by the direct method are identical with the 24 that are not rejected as invalid by the indirect method. It follows that the two methods must be used as supplementary to one another. For, apart from the use of the indirect method we should not have proved that the moods established as valid were the *only* valid moods; and apart from the use of the direct method we should not have proved that the moods unrejected as invalid were themselves valid. In short, by the direct method we establish the conditions

that are *sufficient* to ensure validity, and by the indirect those that are *necessary* to ensure validity.

§ 15. The attached diagram, taking the place of the mnemonic verses, indicates which moods are valid, and which are common to different figures. The squares are so arranged that the rules for the first, second and third figures also show the compartments into which each



mood is to be placed, according as its major, minor or conclusion is universal or particular, affirmative or negative. The valid moods of the fourth figure occupy the central horizontal line.

§ 16. A very simple extension of the syllogism and of the corresponding antilogism is treated in ordinary logic under the name *Sorites*, which is a form of argument comprised of propositions forming a closed chain;

and may be defined as 'an argument containing any number of terms and an equal number of propositions, such that each term occurs twice and is linked in one proposition with one term, and in another with a different term.' E.g. an argument of this form, containing five terms, would be represented by the five propositions:  $a-b$ ,  $b-c$ ,  $c-d$ ,  $d-e$ ,  $e-a$ , where each term placed first may stand indifferently either for subject or for predicate. Now it will be found that the necessary and sufficient rules for inferences of this form are virtually the same as for the three-termed argument; viz.

A. *Rules of Quality.*

- ( $a_1$ ) For an affirmative conclusion, all the premisses must be affirmative.
- ( $a_2$ ) For a negative conclusion, all but one of the premisses must be affirmative.

B. *Rules of Distribution.*

- ( $b_1$ ) Any term recurring in the premisses must be distributed in (at least) one of its occurrences.
- ( $b_2$ ) Any term occurring in the conclusion must be undistributed, if it was undistributed in its premiss.

The rules for the corresponding antilogism reduce to two, viz.

- A. *Rule of Quality:* All but one of the propositions must be affirmative.
- B. *Rule of Distribution:* Every term must be distributed at least once in its two occurrences.

§ 17. There are certain irregular forms of syllogism or of sorites, which may be reduced to strict syllogistic form by the employment of certain logical principles, the nature of which we shall proceed to discuss. The

arguments to be considered are those which involve a larger number of terms than of propositions; and it is necessary, in order to test the validity or invalidity of such arguments, to substitute if possible, for one or more of the propositions, an equivalent proposition, which will diminish the terms to the number of propositions. This is done by means of obversion, conversion, and other logical modes. Until this substitution is made, the argument may be valid, and yet break one or more of the rules of syllogism. Thus two of the premisses may be negative, and the argument yet be valid, the apparent violation of the rule being due to the presence of more than the proper number of terms; for example,

No right action is inexpedient,  
This is not a wrong action,  
∴ This is expedient.

Here by merely obverting the two premisses we arrive at the standard syllogism of the first figure, namely:

Every right action is expedient,  
This is a right action,  
∴ This is expedient.

In all cases of substituting for a proposition some equivalent, we may require, besides simple conversion, the replacement of some term by one of its cognates. Thus in obversion, we replace  $P$  by not- $P$  or conversely;  $P$  and not- $P$  being the simplest case of cognates. Again any relative term may be replaced by its cognate correlative. Now in the previous illustration obversion alone was required, whereas if the major had been written 'No inexpedient action is right' conversion would have been required before obverting. To illustrate the



replacement of a relative by its correlative, we may take the old example from the Port Royal logic,

The Persians worship the sun,  
The sun is a thing insensible,  
∴ The Persians worship a thing insensible.

This argument contains five terms, viz., the Persians, worshippers of the sun, the sun, a thing insensible, and worshippers of a thing insensible. The process of reducing this argument to a strictly three-termed argument is effected by what is called 'relatively converting' the major premiss, and again 'relatively converting' the conclusion syllogistically arrived at from our new premisses. The transformed argument then assumes the form of a strict syllogism in the third figure:

The sun is worshipped by the Persians,  
The sun is a thing insensible,  
∴ A thing insensible is worshipped by the Persians,  
where, by converting the conclusion, we reach that required,

The Persians worship a thing insensible.

§ 18. The question whether the syllogism is actually used in thought process is met by noting that, while in ordinary discourse it is rare to find three propositions constituting a syllogism explicitly propounded, arguments of a syllogistic nature are of frequent occurrence. These syllogisms are expressed as enthymemes, i.e. with the omission of one at least of the requisite propositions. Now in an enthymeme there is one, and only one, proposition which could be introduced to render the corresponding syllogism valid. For this reason the enthymeme is liable to one or other of two forms of attack: first it may be attacked on the ground that the premiss

supplied by the hearer is true, and yet renders the argument invalid; or secondly, that the premiss supplied by the hearer is false, and is yet the only one which would render the argument valid. The former case would be said to involve a formal fallacy; and the latter a material fallacy. For example: Consider the enthymeme, 'This flower is a labiate, because it is square-stalked.' Here the premiss 'All labiates are square-stalked,' which is true, renders the argument formally invalid; on the other hand, the proposition which renders the argument formally valid, namely 'All square-stalked plants are labiates' is false. These fallacies arise, for the most part, in the case of disagreement between disputants with respect to the conclusion. An enthymeme is free from both kinds of fallacy when the premiss to be supplied is known or accepted by all parties, and at the same time, renders the argument formally valid. Thus a strictly valid argument is expressed in the form of an enthymeme when there is no question with regard to the truth of the omitted proposition which will render the argument formally valid.

This may be instructively illustrated by taking examples where either the major or the minor premiss in each of the first three figures of syllogism is omitted.

'Mr X is a profiteer, and therefore he ought to be super-taxed,'

this argument is acceptable on condition that the required major premiss 'All profiteers ought to be super-taxed' is admitted.

'All bodies attract, therefore the earth attracts,'

this requires the minor premiss 'The earth is a body.'

'Mr X ought not to be super-taxed, therefore he is not a profiteer,'

this requires the same major premiss as in the first example.

'Everyone present voted for Home Rule, therefore Mr Carson was not present.'

This requires as minor premiss 'Mr Carson would have voted against Home Rule.'

'Mr Carson was present, therefore someone there must have voted against Home Rule.'

This requires the major premiss 'Mr Carson would vote against Home Rule.'

'The earth is not self-luminous, therefore not all attracting bodies are self-luminous.'

This requires as minor premiss 'The earth is an attracting body.' These three pairs of arguments are respectively in the first, second and third figure of syllogism.

§ 19. Having restricted my technical treatment of the syllogism to a single chapter, it will be easily inferred that I attach considerable importance to this form of inference, while at the same time I hold it to be only one among many other equally important forms of demonstrative deduction. Syllogism is *practically* important because it represents the form in which persons unschooled in logical technique are continually arguing. It is *theoretically* important because it exhibits in their simplest guise the fundamental principles which underly all demonstration whether deductive or inductive. It is *educationally* important because the establishment of its valid moods and the systematisation and co-ordination of its rules afford an exercise of thought not inferior and in some respects superior to that afforded by elementary mathematics.

## CHAPTER V

### FUNCTIONAL EXTENSION OF THE SYLLOGISM

§ 1. THE categorical syllogism treated in the last chapter is correctly described as subsumptive. This term applies strictly to the first figure alone—which may be called the *direct* subsumptive figure, and since, either by antilogism or by conversion, the other figures can be reduced to the first, these may be called *indirectly* subsumptive figures. As explained in Chapter I, this form of inference employs in the most simple manner the Applicative followed by the Implicative principle. The ordinary subsumptive syllogism has a conclusion applying to the same range as the instantial minor, and its typical form is:

Everything is  $p$  if  $m$  ;  
This is  $m$  ;  
∴ This is  $p$ .

The first step in the extension of the ordinary syllogism to its functional form is to take a conjunction of disconnected syllogisms of the type:

'Everything is  $p$  if  $m$  ; This is  $m$  ; ∴ This is  $p$ .'  
'Everything is  $p'$  if  $m'$  ; This is  $m'$  ; ∴ This is  $p'$ .'  
'Everything is  $p''$  if  $m''$  ; This is  $m''$  ; ∴ This is  $p''$ ,' etc.

We next take  $m, m', m''$ , etc., to be determinates under the determinable  $M$ , and  $p, p', p''$ , etc., to be determinates under the determinable  $P$ . If then we can collect these major premisses into a general formula holding for every value of  $M$  and  $P$  in accordance with

the mathematical equation  $P=f(M)$ , then we have an example of what may be called the functional extension of the syllogism, or (more shortly) of the functional syllogism, where the major or supreme premiss may be expressed in the simple form  $P=f(M)$ . Thus in the subsumptive syllogism the terms that occur in the minor and major premisses are merely repeated in the conclusion; but, in the functional syllogism which yields an indefinite number of different conclusions for the different minors, the terms which occur in these different minors and conclusions are specific values of the determinables presented in the supreme premiss. Now it will be seen that no other principles are used in the functional syllogism, except the Applicative and Implicative, which together are sufficient to extend deduction beyond the scope of merely subsumptive syllogism. As a concrete example, let us take the formula of gravitation, which may be elliptically expressed 'Acceleration  $P$  varies inversely as the square of the distance  $M$ ,' and written in the form  $P=\frac{1}{M^2}$ .

Then, by the Applicative Principle:

'If  $M=7$ , then  $P=\frac{1}{49}$ ,'

and adding, as Minor Premiss:

'In this instance  $M=7$ ,'

we infer, by the Implicative Principle:

'In this instance  $P=\frac{1}{49}$ .'

Similarly when the value of  $M$  is 11, the value of  $P$  will be  $\frac{1}{121}$ , and so on. The same form of inference holds for two or more independent variables: thus

Boyle's Law may be written ' $T=239 PV$ '; then, as before, we infer:

When the value of  $P$  is 5, and the value of  $V$  is 2, the value of  $T$  will be 2390.

When the value of  $P$  is 3, and the value of  $V$  is 7, the value of  $T$  will be 5019.

§ 2. A functional expression is of course familiar to the mathematician, but it will be important to examine the logical principles in accordance with which a universal functional formula operates in mathematical demonstration. In the first place we may observe that, as in ordinary syllogism, the supreme or major functional universal must always have been ultimately established by means of inductive generalisation, and in the last resort from intuitive or experiential data. Further, the functional universal may be said to be a universal of the second order, because it not only universalises over every *instance* of a given value  $m$ , but applies also to every *value* of  $M$ . In deducing from the major ' $P=f(M)$ ' conjoined with the minor 'A certain given instance is  $m$ ,' we reach the conclusion 'This given instance is  $p$ ,' where  $p$  is found from the equation ' $p=f(m)$ .' Here it is to be observed: first, that this type of conclusion can be drawn, not only for the minor which predicates  $m$ , but also for minors which predicate any other value of  $M$ ; and secondly, that the character predicated in each conclusion is not merely what is predicated in the functional major, but a determinate specification of this predicate.

In the functional syllogisms that we shall consider in this chapter, the functional major is to be understood to express a factual rule, or more particularly a Law of

Nature. The general conception of a Law of Nature has been discussed (in Chapter XIV of Part I) under the head of the Principles of Connectional Determination. There it is shown that a typical uniformity or Law of Nature may be expressed in the form that the variations of a certain phenomenal character depend upon an enumerable set of other phenomenal characters; of these the former is taken to be connectionally dependent upon the others, which are connectionally independent of one another. A specific universal, which expresses such a relation of dependence may also be called a Law of Covariation; for the nature of the dependence (say) of  $P$  upon  $ABC$  is such that all the possible variations of which  $P$  is capable are determined by the joint possible variations of  $A, B, C$ , which are themselves connectionally independent of one another.

§ 3. We have a special case of this relation of dependence or covariation when the determined character can be represented as a mathematical function of the determining characters; and it is this special case which gives rise to the functional syllogism. Now, in a functional major expressed (say) in the form  $P=f(A, B, C)$ , it may in general be assumed that the correlation of these variables is such that, not only can the value of  $P$  be calculated from any assigned values of  $A, B, C$ ; but also, conversely, that the value of  $A$  can be calculated from the values of  $P, B, C$ ; and that of  $B$  from the values of  $P, A, C$ ; and that of  $C$  from the values of  $P, A, B$ ; and similarly for a larger number of such connected variables. This process is expressed in mathematical terms as solving the equation  $P=f(A, B, C)$ , to find the value of  $A$ , which is thus calculated as a

certain function of  $P, B, C$ , and so on. A convenient symbolisation for these several equations will be as follows:

$$P=f_p(A, B, C):$$

from which we calculate

$$A=f_a(P, B, C); B=f_b(P, A, C); C=f_c(P, A, B).$$

It is here assumed not only that  $P$  (in mathematical phraseology) is a single-valued function of  $A, B, C$ , but also that in solving this equation to determine  $A$  or  $B$  or  $C$  respectively, these also are single-valued functions of the remaining variables. When this assumption holds, we may speak in a special sense of the *reversibility* of cause and effect; i.e. not only is the effect  $P$  uniquely determined by the conjunction of the cause-factors  $A, B, C$ ; but also each of the cause-factors themselves, such as  $A$ , is uniquely determined by the effect-factor  $P$  in conjunction with the remaining cause-factors  $B$  and  $C$ . In the simplest cases reversibility follows immediately from the *form* of the function as seen in the example given of Boyle's Law. Here we have a correlation between temperature, pressure, and volume, in which a constant, say  $k$ , is involved, and which assumes indifferently the form:

$$\theta = \frac{pv}{k}, \text{ or } v = \frac{k\theta}{p}, \text{ or } p = \frac{k\theta}{v}.$$

In this simple case, the multiplier  $k$  indicates the special form of the function which in the *general* case was represented by the unassigned but constant symbol  $f$ . An equation which, in this way, solves uniquely for all the variables is known as linear. But even in the case of non-linear equations we must be able to deter-

mine, amongst the theoretically possible solutions for any one of the variables, that which is the sole factual value. In other words, a unique determination of all the variables, in terms of a given number of them, may be taken as expressing the actual concrete fact.

§ 4. In this connection it is important to note the *number* of variables entering into the functional formula. In Boyle's Law this number is three; i.e. there are three variables, any one of which is connectionally dependent upon the two remaining variables, so that the scope of dependence may be measured either as two or as three: for the functional formula contains three variables which are *notionally* independent of one another, namely  $p, v, \theta$ ; but of these two only are *connectionally* independent of one another. These two may be taken indifferently either as  $v$  and  $\theta$ , or as  $p$  and  $\theta$ , or as  $p$  and  $v$ , where, according to the alternative taken,  $p$  or  $v$  or  $\theta$  is connectionally dependent upon the two others. In general, when there are  $r$  functional relations, connecting  $n$  notionally independent variables, then any  $n-r$  of these can be taken as connectionally independent of one another, and each of the remaining  $r$  as connectionally dependent jointly upon the others.

Thus when  $n=8$ , and  $r=3$ , the three functional relations may be symbolised:

$p=f_p(a, b, c, d, e)$ ;  $q=f_q(a, b, c, d, e)$ ;  $r=f_r(a, b, c, d, e)$ ; or adopting a shorter notation:

$$p=f_p.abcde; q=f_q.abcde; r=f_r.abcde.$$

Such a trio of equations are taken to be implicationally independent of one another; i.e. from neither one or

two of them could we infer the third. Otherwise the number three would reduce to two. Now the number of implicationally independent equations is necessarily the same as the number of connectionally dependent variables. Hence for the case under consideration we may express the three independent functional relations in either of four typical forms:

- |                    |                 |                 |
|--------------------|-----------------|-----------------|
| 1. $p=f_p.abcde$ ; | $q=f_q.abcde$ ; | $r=f_r.abcde$ . |
| 2. $a=f_a.pbcde$ ; | $q=f_q.pbcde$ ; | $r=f_r.pbcde$ . |
| 3. $p=f_p.abcqr$ ; | $d=f_d.abcqr$ ; | $e=f_e.abcqr$ . |
| 4. $a=f_a.pqrde$ ; | $b=f_b.pqrde$ ; | $c=f_c.pqrde$ . |

The first trio expresses the three effect-factors separately in terms of the five cause-factors jointly; the second expresses one cause-factor and two effect-factors separately in terms of one effect-factor and four cause-factors jointly; the third expresses one effect-factor and two cause-factors separately in terms of three cause-factors and two effect-factors jointly; and the fourth expresses three cause-factors separately in terms of three effect-factors and two cause-factors jointly.

In illustration of this general principle we will consider the Law of Gravitation, which may be formulated

$$A=c \frac{m_1 m_2}{d^2},$$

where  $A$  is the force of attraction of any two masses  $m_1$  and  $m_2$  whose distance is  $d$ ,  $c$  being constant for all variations of  $m_1$  and  $m_2$ , as well as of  $d$ . In any application of the above formula we must first suppose  $m_1$  and  $m_2$  to be constant, so that the variation of  $A$  depends solely upon that of  $d$ . The algebraical equation here is, however, logically incomplete. In the first



place, as regards the effect  $A$ , we must add the statement that it is a force acting in the direction of the line joining  $m_1, m_2$ . In the second place, as regards the cause  $d$ , not only must the *distance* of the line joining  $m_1$  and  $m_2$  be taken as a cause-factor, but also its direction.

In comparing the Law of Gravitation with Boyle's Law, the constants  $k, c, m_1, m_2$ , represent unchangeable properties of the bodies concerned, while  $p, \theta, v, d$  represent their changeable states or relations. It is necessary then to include amongst the independent cause-factors the *permanent* properties of bodies as well as their alterable states or relations.

§ 5. In our typical expression of a set of functional equations the number of variables taken to be connectionally independent was the *same* in all the several equations. But a very important type of connectional formulae is that in which equations enter involving *different* numbers of independent variables. Consider the following:

Let a body be allowed to fall *in vacuo*. Here the two independent cause-factors are the mass ( $m$ ) of the body and the distance ( $d$ ) from which it falls to the earth. The effect-factors to be considered are the time ( $t$ ) of falling, and the impulse ( $p$ ) of the body upon the earth. Since out of the four variables  $m, d, p, t$ , two of them, namely  $m$  and  $d$ , are (as cause-factors) connectionally independent, the standard form in which *both* of these would enter into the function is

$$p = f_p \cdot md \text{ and } t = f_t \cdot md.$$

But, where the body falls *in vacuo*, the time ( $t$ ) is in-

dependent of the mass ( $m$ )<sup>1</sup>. Hence in this case the two formulae assume the form

$$p = f_p \cdot d \text{ and } t = f_t \cdot d.$$

In this case, since the solution of the equations gives uniquely determined roots, we have:

$$(1) p = f_p \cdot md, (2) m = f_m \cdot pd, (3) d = f_d \cdot mp$$

and

$$(4) t = f_t \cdot d, (5) d = f_d(t),$$

and, by substitution from (5) in (1), (2), (3) respectively,

$$(6) p = f_p \cdot mt, (7) m = f_m \cdot pt, (8) t = f_t \cdot mp.$$

Now, since, of the four variables  $m, d, p, t$ , any two except  $t$  and  $d$  may be taken as connectionally independent, either one of the following pairs of connectional equations may be used, thus:

|                                                                             |             |   |                                         |
|-----------------------------------------------------------------------------|-------------|---|-----------------------------------------|
| Taking $m$ and $d$ as independents: $p = f \cdot md$ with $t = f \cdot d$ , |             |   |                                         |
| "                                                                           | $m$ and $t$ | " | : $p = f \cdot mt$ " $d = f \cdot t$ ,  |
| "                                                                           | $p$ and $d$ | " | : $m = f \cdot pd$ " $t = f \cdot d$ ,  |
| "                                                                           | $p$ and $t$ | " | : $m = f \cdot pt$ " $d = f \cdot t$ ,  |
| "                                                                           | $m$ and $p$ | " | : $d = f \cdot mp$ " $t = f \cdot mp$ . |

Giving to the unassigned functions their actual form, we have here

$$(6) p = mgt \text{ and } (5) d = \frac{1}{2}gt^2,$$

where the constant  $g$  stands for the acceleration 32 ft. per second. Solving these equations so as to express the effect-factors  $p$  and  $t$  in terms of the cause-factors  $m$  and  $d$ , we have

$$(1) p = m\sqrt{2gd} \text{ and } (4) t = \sqrt{\left(\frac{2d}{g}\right)}.$$

§ 6. The example just given suggests a certain further characteristic of the connectional equations

<sup>1</sup> This illustrates the principle underlying the inductive method of agreement; where  $m$  is eliminated as a cause-factor relative to the effect  $t$ , since a variation in  $m$  does not entail a variation in  $t$ .



of applied mathematics. As will be seen in the above illustration, the connectional equations from which the deductive process derives other but equivalent equations are of a mixed nature as regards the variables that are taken as independents. Of the two equations  $p = mgt$ , and  $d = \frac{1}{2}gt^2$ , from which the other equations are derived, the former expresses the effect-factor  $p$  in terms of the cause-factor  $m$  and the other effect-factor  $t$ ; while the second expresses the cause-factor  $d$  in terms of the effect-factor  $t$ . It is therefore necessary to solve this pair of equations by an appropriate process in order to derive the pair of equations which express the effect-factors in terms of the cause-factors; namely in the form

$$p = m\sqrt{2gd}, \quad t = \sqrt{\left(\frac{2d}{g}\right)}.$$

What holds in this particular example may be generalised. Instead of separating variables that are given from those which have to be deduced, we have a set of equations (corresponding in number to the *dependent* variables) which all the variables taken together have to satisfy. Thus, in the above example, the equations were not at first expressed by taking a pair of cause-factors as independent upon which the pair of effect-factors depended, but the first of the two equations was taken from the pair in which  $m$  and  $t$  were supposed to be independent, and the other from the pair in which  $m$  and  $t$ , or  $p$  and  $t$  were taken as independent. That the particular example of the falling body, originally taken to illustrate a different principle, should have lent itself to the principle now under consideration, is more or less accidental, and we will now put

forward an example which more naturally exhibits this new speciality of a set of determining equations.

Thus: consider the effect of mixing two substances at different temperatures in order to find the resultant temperature which will be reached when thermal equilibrium has been established. Here we must take as the causally determining factors, the two initial temperatures  $\theta_1$ ,  $\theta_2$ , and the two thermal capacities of the substances  $k_1$  and  $k_2$ . The factors to be determined are the heat  $H_1$  entering into or passing from the one substance, and the heat  $H_2$  passing from or entering into the other, together with the final temperature  $\theta$ . Now the equations that must be here used express the conditions that are to be satisfied—the effects not being, in the first instance, expressible as functions of the cause-factors. These equations of condition are the following three:

$$H_1 = k_1(\theta - \theta_1); \quad H_2 = k_2(\theta - \theta_2); \quad H_1 + H_2 = 0;$$

from which we find

$$\theta = \frac{k_1\theta_1 + k_2\theta_2}{k_1 + k_2};$$

$$H_1 = \frac{k_1k_2(\theta_2 - \theta_1)}{k_1 + k_2}; \quad H_2 = \frac{k_1k_2(\theta_1 - \theta_2)}{k_1 + k_2}.$$

The solutions of these equations give the three values  $H_1$ ,  $H_2$ , and  $\theta$  (respectively) that were to be determined. Thus, in the final solution we have succeeded in expressing the factors to be determined in terms of the determining factors. But, in the equations expressing the conditions to be satisfied, the first two express an effect-factor as a function of two of the cause-factors and one of the effect-factors, and the third equation

expresses one effect-factor as a function of another effect-factor. The use of equations of this kind is necessitated by the inadequacy of our knowledge of the precise temporal process by which the causal conditions operate until the final issue is reached. Thus, in the actual process, heat will be passing to and fro from one to the other of the two substances, and this will entail a rise or fall of their temperatures in an incalculable way, which may be roughly expressed by suggesting that the quantum of heat entering the cooler body may be too great, so that the flow of heat will immediately be reversed; and this process might be conceived as involving even an infinite number of ingoings and outgoings of heat. What we know, however, is that at any stage of the process the heat that leaves one body must be equal to the heat that enters the other, whether this quantum is to be reversed in the next stage or not. It is this law which is expressed in our third equation, while the other two equations express a law or property, specific to the two substances, which correlates the effect upon the temperature with the quantum of heat which enters or leaves the body. What then we know, are these conditions of conservation of the total heat, and the several thermal capacities of the bodies, and from this knowledge the final effects can be calculated. It would appear, in fact, that the cases in which this logical principle is exhibited are those in which we know what is entailed in a final state of equilibrium, without having adequate knowledge for tracing in detail the perhaps oscillating processes which take place in the lapse of time before the final state of equilibrium is reached.

We have illustrated this in the simplest case, where only two substances are mixed, but the reader will easily be able to construct the corresponding equations for any given number of different substances. In all cases the final or resultant temperature is equivalent to the arithmetic mean of the initial temperatures, each 'weighted' by the corresponding thermal capacity. Thus

$$\theta = \frac{k_1\theta_1 + k_2\theta_2 + \dots + k_r\theta_r}{k_1 + k_2 + \dots + k_r}.$$

Now having given an illustration from physics, we will give a closely analogous illustration from economics. The formula of covariation which connects the *quantity* of a commodity that is exchanged with its *price* is such that the two opposed parties shall be satisfied at the rate of exchange finally agreed upon. Now the formula of covariation on the side of demand is assumed to be connectionally independent of that on the side of supply. That which represents the economic attitude of the consumers depends solely upon their relative desires for different commodities, their monetary resources, and—we may add—the prices at which they are able to buy commodities other than that under consideration. In the same way, the attitude of the producers is wholly independent of that of the consumers; and depends upon the contract-prices current for the employment of the several agents of production, and upon the efficiency of these agents when co-operating in producing the commodity. It will thus be seen that the several factors that determine the conditions of supply are independent of those that determine the conditions of demand. Here, as in the case of thermal equilibrium,

the equations of condition express, not the effect-factors as functions of the cause-factors, but the conditions taken together which satisfy the consumers and the producers regarded each as economically independent of the other.

This economic illustration differs from the case of thermal equilibrium in the important respect that the two functions of demand and supply respectively replace the actually operating cause-factors, which are highly complex and do not explicitly enter into the equations to be solved.

§ 7. The above illustrations of the functional extension of the syllogism have shown how, by the use of a set of functional premisses standing as majors, we may take not only minors which enable us to infer an effect-factor from the knowledge of a given cause-factor, but also minors which enable us to infer a cause-factor from the knowledge of a given effect-factor. The supposition upon which this is based has been called the Principle of Reversibility. We shall now show that it is this principle which underlies the so-called method of Residues, and other similar deductive processes. The canon of this method is stated by Mill as follows:

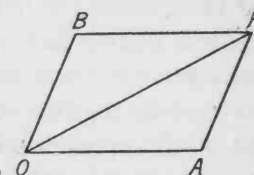
'Subduct from any phenomenon such part as is known by previous inductions to be the effect of certain antecedents, and the residue of the phenomenon is the effect of the remaining antecedents.'

In using the term 'subduct' Mill intends no doubt to hint that, in the simplest cases, for 'subduct' we may substitute 'subtract.' Thus Jevons, in his *Elementary Lessons*, takes the case of 'ascertaining the exact weight of any commodity in a cart by weighing the cart and

load, and then subtracting the weight of the cart alone, which has been previously ascertained.' Here what corresponds to the effect ( $p$ ) is the weight of the cart and load together, and its causes are ( $a$ ) the weight of the commodity, and ( $b$ ) the weight of the cart: so that the functional datum assumes its simplest form, viz.  $p = a + b$ , which by reversibility gives  $a = p - b$ . This is a case of solving an equation  $p = f(a, b)$  to find  $a$ , and deducing  $a = f(b, p)$ , the equations being linear. The next simplest example of such reversibility is that of the composition of forces. Here the

diagonal  $OP$  represents the effect, and the sides  $OA, OB$ , the cause-factors. Just as, given the two cause-factors  $OA$  and  $OB$ , we draw  $AP$  parallel and equal to  $OB$  to find the effect  $OP$ ; so, given  $OA$  as one cause-factor and  $OP$  as effect, we may draw  $OB$  parallel and equal to  $AP$  to find the other cause-factor  $OB$ . Innumerable other examples may be given of reversibility for more or less complicated cases. But the classical example most frequently cited is the Adams-Leverier discovery of the planet Neptune from the observed movements of Uranus. Here we may represent the positions and masses of the Sun, of the Moon and of the unknown Neptune by the symbols  $a, b, c$  respectively; and the movement of Uranus by the symbol  $p$ . Thus  $p$  was theoretically known as a given function of  $a, b, c$ , say  $p = f_p(a, b, c)$ , where  $p$  stands elliptically for the effect, and  $a, b, c$  for the several cause-factors. The solution for  $c$  was then uniquely calculated in the form

$$c = f_c(p, a, b).$$



Now it will be observed that the so-called Method of Residues, which is based upon the assumption of reversibility is purely deductive, in that (1) it employs only the Applicative and Implicative principles of inference, and (2) the conclusion obtained applies solely to the specific instances for which the calculation is made. This consideration shows that there is no justification for putting Herschel's method of Residues under the head of methods of induction, along with such methods as those of Agreement and Difference; for, on the grounds above alleged, it is purely deductive. On this matter Mill sees half the truth; for, in comparing the Method of Residues with that of Difference, he remarks that the negative instance in the former is not the direct result of observation, but has been arrived at by deduction. And again, in his formulation of the Canon of Residues, he speaks of 'such part of the phenomenon as is known by previous induction,' where he fails to note that what is *known by previous induction* functions merely like the major premiss of a syllogism, and therefore does not in any way render the inference inductive. What holds for the method of Residues holds also of many less technical processes which, while purely deductive, have been obscurely conceived as inductive. For instance, the procedure in a judicial enquiry or by a police detective or of historical research in discovering the specific cause of a complicated set of circumstances constituting an observed effect, is purely deductive; for it employs as major premiss known laws of human or physical nature under which the known circumstances are to be subsumed in the minor; while the conclusion refers solely to the case *sub judice*.

§ 8. Something should be said in explanation of the fact that inferences of this kind are so frequently spoken of as inductive. It is not only because the major premiss must itself have been obtained by induction, but further because the minor premiss represents a fact obtained by observation, that logicians have made this mistake; for the notion of observation or experimentation as the method by which new knowledge is acquired is invariably associated with induction. But it should be pointed out that there is here a confusion between the matter and the form of an inference. Mere syllogism will obviously yield new material knowledge, provided that the minor premiss represents new material knowledge such as can only be obtained by observation. For example, from the observation that the importation of food has been taxed, we may infer the new material knowledge that the price of food will rise at a certain time in a certain economic market, if we have been otherwise assured of the major premiss appropriate to the circumstance. The *form* of such an inference is purely deductive, and the fact that historical research—and not a merely foreknown universal formula—has been required to establish the minor does not render the argument in any sense inductive; for the conclusion holds only of the period and region to which the causal occurrence which has been discovered applies, and does not involve any inductive generalisation from one period or region to others. A further explanation of this common error is to be found in the fact that the conclusion reached deductively for a given instance may often be *verified* by awaiting the occasion for observing the effect in that instance. Now this process of verification

merely assures us that we have adequately estimated the causes operating in the given instance; but it has been almost invariably confused with the process of verifying, or rather *confirming*, the major premiss itself regarded as a problematic hypothesis as yet unproven.

§ 9. We ought now to distinguish, in these functional extensions of the syllogism, the element which is purely subsumptive from that which is functional; for the two elements are practically always united in any concrete inference of the functional kind. It will be found that the factual formulae used in applied mathematics as major premisses for deduction necessarily involve two kinds of constituent, one of which is known as variable and the other as constant. The mathematical use of the term *constant* presents certain difficulties from the logical point of view. There are certain constants—e.g. the specific integers and the algebraical operators—which are absolutely constant in the sense that in all their occurrences they stand for the same thing and are entirely independent of context. But those so-called constants which are dependent upon context are only referentially constant, being actually variable in precisely the same sense as the symbols that mathematicians recognise as variable. To explain this we may select illustrations from an innumerable variety of formulae used in applied mathematics. Consider, for instance, the formula which expresses the elasticity of a solid body which can support tension. The rule upon which the extension of such a body depends is shortly expressed in the formula  $T = kE$ , where  $T$  stands for the variable tension and  $E$  for the variable extension; while  $k$ , which is said to be constant, stands for the elasticity of

the particular kind of solid for which the rule holds. Now this coefficient of elasticity, though constant for all possible variations of extension and tension of the body, yet varies from one kind of solid substance to another. We shall show, then, that such a coefficient, which mathematicians call constant, is used in the deductive process *subsumptively*, while that which is explicitly regarded as variable is used *functionally*. We may mark the real variability of a so-called constant by a subscript indicating the specific kind of substance of which the coefficient can be predicated. Thus  $k_s$  will stand for the coefficient of elasticity of the kind of substance named  $s$ ; while  $k_{s'}$  (say) will stand for that holding for the kind of substance called  $s'$ . To express the mathematical procedure in strictly explicit logical form:

*Major Premiss.* Every body, say  $b$ , which is  $k_s$  has the property expressed by the algebraical equation  $T = k_s E$ .

*Minor Premiss.* A certain body  $b$  is  $k_s$ .

*Conclusion.* The body  $b$  has the property expressed by the equation  $T = k_s E$ .

Now this is a merely subsumptive syllogism, in which the coefficient  $k_s$  and the body  $b$  recur unmodified in the conclusion as in the premisses. Thus, the coefficient which is called constant is used solely in a subsumptive form of syllogism; but, inasmuch as a similar formula applies to bodies of a different nature (such as  $s'$ ), the coefficient  $k$  is not absolutely constant but varies according to the substance of the solid. In logical analysis, we must recognise the distinct ways in which the so-called constants and the so-called variables enter into the deductive process. This may be expressed



logically by defining the *order* in which the variations have to be made. For we have *first* to consider variations of the so-called variables, which determine the range of the conclusion as holding for every case over which the constant applies. Only *after* this range of variation has been taken into consideration may we proceed to vary the so-called constants, and for any new value carry out the same range of variations of the variables. In language borrowed from mathematical terminology, we may say that the variations of the explicit variables are to be made within the bracket, while the variations to be made of the so-called constants are to be made outside the bracket.

## CHAPTER VI

### FUNCTIONAL DEDUCTION

§ 1. UNDER this heading we shall discuss the principles underlying the deduction of formulae in the sciences of mathematics and logic. Although properly speaking pure mathematics is a development of logic, yet certain important points of distinction between the two sciences must be brought out. It has been very commonly assumed that the sole method of deductive procedure in pure mathematics, including Geometry, is syllogistic. Now although it will be found that no fundamental principle is employed in mathematical deduction other than the Applicative—which is essential for syllogism—yet the conclusions successively derived from previously established formulae are not such as could be inferred by means of any mere chain of syllogisms. To explain this, it is necessary to point out the peculiar nature of the relation between conclusion and premisses in mathematical processes. Ordinary syllogism, as has been explained, is of the comparatively simple type denominated subsumptive. If subsumptive inferences only were used in algebra or geometry, it would be impossible to demonstrate conclusions except for special cases subsumable under the primary intuited axioms or under some previously established formulae. Thus from such premisses as: 'Everything that is *m* is *p*' and 'Everything that is *n* is *q*' we could infer sub-



sumptively only that 'Everything that is  $m$  and  $n$  is  $p$  and  $q$ .' In other words, by means of subsumptive deduction, we can infer only that what holds universally of the members of a genus,  $m$  or  $n$ , holds universally of the members of their common species, viz. of the things that are characterised as being both  $m$  and  $n$ . For example: in geometry, having established a formula for all triangles and a formula for all right-angled figures, we could by merely subsumptive inference predicate of any species of triangles—say right-angled—only what could be predicated of all triangles; and similarly we could predicate of any species of right-angled figures—say three-sided—only what could be predicated of all right-angled figures. But actually in geometry we prove a property (viz. the Pythagorean) of all right-angled triangles which is not the same as any universal property either of three-sided or of right-angled figures. Similarly in algebra, we can deduce properties of all integers divisible by 2 and divisible by 3, which hold neither of all integers divisible by 2 nor of all integers divisible by 3. A predicate which holds for all members of a species, but not for all members of any genus to which by definition the species belongs, is technically known as a proprium or ἰδιον, either of which term may be translated *property*. It is one of the special objects of this chapter to analyse the process by which properties, in this technical sense, are deduced. It will be shown that, in the deductions peculiar to pure mathematics, the premisses and conclusions assume the form of functional equations; and that it is owing to this characteristic that properties in the technical sense can be deductively demonstrated. We therefore give the

name *functional deduction*, in antithesis to subsumptive or syllogistic deduction, to the specifically mathematical form of inference.

§ 2. Before entering upon the main discussion it will be well further to consider the nature of the Aristotelean ἰδιον. Many modern logicians have failed to grasp the important significance to be attached to this notion. Elementary textbooks, such as that of Jevons, define a property of a class as any character not included in the connotation, which can be predicated of *all*, as distinct from an accident which can be predicated only of *some*, members of the class. On the other hand, Mill attempts to define a proprium in closer connection with the scholastic development of Aristotle's doctrine, and distinguishes not merely between an invariable and a variable predicate of a class—which satisfies Jevons—but defines a proprium as a predicate not included in the connotation of the class (and therefore assertible in a proposition not merely verbal) but following necessarily from the connotation alone. But since a proposition which merely asserts connotation is verbal, this account of the proprium is incompatible with the theory—so clearly expounded in his chapters on Definition and on Verbal Propositions—that no conclusion can be drawn from merely verbal propositions that is not itself merely verbal. From this it follows that in order demonstratively to establish any invariable character that can be regarded as necessary, we require as premisses not only definitions but also real or genuine propositions, and, in mathematics, ultimately axioms. It is true that Mill distinguishes two ways in which the proprium may follow necessarily from the connotation:

'it may follow as a conclusion follows premisses, or it may follow as an effect follows a cause.' But this distinction is purely illusory and wholly irrelevant to the notion of necessity of demonstration; for, in both cases, the ground for Mill's account of a proprium as necessarily following from the connotation is that appropriate knowledge will enable us to infer demonstratively the proprium from the connotation. A legitimate distinction may be drawn according as the major premiss from which a proprium is inferred is of the nature of an axiom or of a causal law. Indeed Mill himself goes on to say that the necessity attributed to the proprium means that 'its not following would be inconsistent with'—i.e. *its following could be inferred from*—either an Axiom or a Law of Nature. Thus in both cases the notion of *following* is the same, and simply means *inferrible from*. The proprium, therefore, never follows from the connotation *alone*, but requires in addition one or other of the two species of real propositions, axiomatic or experiential, to serve as major premiss.

§ 3. The functional equations used in the deductions of pure mathematics in some respects differ from and in others agree with those used as major premisses in the process discussed under the head of the functional extension of the syllogism. The equation used in this latter process serves as a single major premiss for a number of specific conclusions found by replacing the variables by their specific values. Here the functional equation assumes the form  $P = f(A, B, C)$  for all values of  $A, B, C$ . But the equations used in the process of functional deduction are of the form  $f(A, B, C) = \phi(A, B, C)$  for all values of  $A, B, C$ , where all the variables are

independently variable, and the equation therefore contains no such symbol as  $P$  that can be exhibited as dependent upon the others. The distinction between these two types of equation is familiar to mathematicians; the former may be called a *limiting*, the latter a *non-limiting* equation. The limiting equation is generally used to determine one or other of the quantities  $P, A, B$ , or  $C$ , in terms of the remainder; so that here we associate the antithesis between dependent and independent with the antithesis between unknown and known; whereas, in the non-limiting equation, no one of the variables can be regarded as unknown and as such expressible in terms of the others regarded as known. The distinctions that have been put forward between these two types of functional process are tantamount to defining the functional syllogism as that which proves factual conclusions from factual premisses, and functional deduction as that which proves formal conclusions or formulae from formal premisses, i.e. from formulae previously established. It will further be observed, from the simple illustrations which follow, that whereas the functional syllogism requires only the one functional equation that serves as major premiss, the process of functional deduction will necessarily involve a conjunction of two or more functional equations, all of which are, as above explained, formal and not factual.

To illustrate the general formula used in functional deduction, viz.:

$$f(a, b, c, \dots) = \phi(a, b, c, \dots)$$

which is understood to hold for every value of the

variables  $A, B, C, \dots$ , we may instance the following elementary examples:

$$(a+b) \times (a-b) = a^2 - b^2$$

and

$$a \times b = b \times a,$$

both of which involve two variables; and again

$$\begin{aligned} (a+b)+c &= a+(b+c) \\ (a+b) \times c &= (a \times c) + (b \times c), \end{aligned}$$

both of which involve three variables. The last three formulae are known respectively as the Commutative, the Associative and the Distributive Law.

§ 4. In the functional equations of mathematics it is important to realise the range of universality covered by any functional formula. This range depends upon the number of independent variables involved in the formula, the range being wider or narrower according as the number of independent variables is larger or smaller. For example, supposing that  $x, y, z$  have respectively 7, 5, 10 possible values; then the number of applications of the formula involving  $x$  alone is 7, that of a formula involving  $x$  and  $y$  alone is 35, and that of a formula involving  $x$  and  $y$  and  $z$  is 350. And in general, the number of applications of a formula is equal to the arithmetical product of the numbers of possible values for the variables involved. Now the number of possible values of any variable occurring in logical or mathematical formulae is infinite; hence, for the cases respectively of 1, 2, 3... variables, the corresponding ranges of application would be  $\infty, \infty^2, \infty^3 \dots$ , constituting a series of continually higher orders of infinity; or rather, in accordance with Cantor's arithmetic, each of the ranges of application for 1, 2, 3... variables is a

proper part of that for its successor, although their cardinal numbers are the same.

Now it will be found that, in inferences of the nature of functional deduction, the derived formula may have a range of application—not narrower than but—equal to or even wider than that from which it is derived. Thus the word *deduction* as here applied does not answer to the usual definition of deduction (illustrated especially in the syllogism) as inference from the generic to the specific; although the only fundamental principle employed in the process is the Applicative, according to which we replace either a variable symbol by one of its determinates or one determinate variant by another. But here a distinction must be made according as the substituted symbol is simple or compound. If we merely replace any one of the simple symbols  $a, b, c$  by some other simple symbol we shall not obtain a really new formula, since the formula is to be interpreted as holding for all substitutable values, and hence it is a matter of indifference whether we express the formula in terms of the symbols  $a, b, c$ , (say) or of  $p, q, r$ . In order to deduce new formulae, it is necessary to *replace two or more simple symbols by connected compounds*.

For those unfamiliar with mathematical methods, it should be pointed out that, when any compound symbol is substituted for a simple, the compound must be enclosed in a bracket or be shown by some device to constitute a single symbolic unit. Though we may always replace in a general formula a simple by a compound symbol, the reverse does not by any means hold without exception. The cases in which such substitution is permissible have been partially explained in the

chapter on Symbolism and Functions. There it was shown that, if a formula involves such compound symbols or sub-constructs as  $f(a, b)$ ,  $f(c, d)$  etc., *and only such*, where none of the simple symbols used in the one bracketed sub-construct occur in any of the others, then these bracketed functions are called disconnected. It is in the case of disconnected functions that free substitutions of simple symbols for the compound are permissible. The reason for this is that, for the notion of a function of any given variants, it is essential that these shall be variable independently of one another. Now, when the different sub-constructs or bracketed functions are connected with one another through identity of some simple symbol, say  $a$ , it is clear that we cannot contemplate a variation of one of these compounds without its involving a variation of the other connected compounds. Hence we should be violating the fundamental principle of independent variability of the variants, if we freely substituted for such connected compounds simple symbols which would have to be understood as capable of independent variation. Hence, it is only when the various compounds involved in a function are unconnected, that for each of such compounds a simple symbol may be substituted.

§ 5. Returning to the problem under immediate consideration, a simple illustration from algebra will show how, by making appropriate substitutions in a given functional formula, we may demonstrate a new formula. Thus, having established the formula that for all values of  $x$  and  $y$

$$(i) \quad (x+y) \times (x-y) = x^2 - y^2$$

we may substitute for  $x$  and  $y$ , respectively, the connected

compounds  $a+b$  and  $a-b$ ; and so deduce (by means of the distributive law for multiplication etc.) that for all values of  $a$  and  $b$ ,

$$(ii) \quad 4ab = (a+b)^2 - (a-b)^2.$$

This is a *new* formula, different from the previous one, because the relation between  $a$  and  $b$  predicated in (ii) is different from the relation between  $x$  and  $y$  predicated in (i). Moreover the range of application for (ii) is no narrower than that for (i); for (i) applies for every diad or couple ' $x$  to  $y$ ,' and (ii) for every diad or couple ' $a$  to  $b$ '; and therefore the ranges for (i) and (ii) are the same. Again, if we have established the Commutative, Associative, and Distributive formulae given above, the reader will see that, by means only of the Applicative principle, we can deduce from these three formulae what is in fact a new formula:

$$(a+b)(c+d) = ac + bc + ad + bd.$$

In this case, the formula deduced has a wider range of application than any of the formulae from which it is deduced. For the premisses for this deduction involve respectively 2, 3 and 3, independent variables, while the conclusion involves 4; showing, as explained in the previous paragraph, that the range of application of the conclusion is wider than that of even the widest premiss. To reach a conclusion inclusive of and wider than the premisses is in general considered the mark of an inductive inference; but we have shown by the above example that, where the premisses are functional formulae involving more than one independent variable, the mere employment of the Applicative principle enables us to reach a formula wider than any of the premisses. Now

it is in accordance with general usage to define deductive inference as that which employs no principles but the Applicative and the Implicative. In the purely deductive process of mathematics, in fact, it is only the Applicative principle that is required; and pure mathematics is regarded as specially typifying the power of mere deduction. It is true, however, that mathematicians have employed a method which involves also the Implicative principle, viz. what has always been known under the name of 'mathematical induction.' In these later days, this method has been regarded as more specifically characteristic of mathematics than any other. But the line of distinction between induction and deduction, in their extended potentialities for demonstrative inference, cannot be drawn on any logical principle that would be universally accepted. It is for this reason that I have attempted to treat in one large division of my *Logic* all varieties of demonstrative inference, on the ground that it is the demonstrative character of these inferences that brings them within one sphere, and that the distinction that might be drawn between deductive and inductive demonstration has no important logical significance comparable with that between demonstrative and problematic inference. Mathematics, as the above adduced inferences illustrate, provides a host of cases in which the Applicative principle alone is explicitly employed without any recourse to the Implicative principle. These inferences might be called purely Applicative<sup>1</sup> in contrast to the syllogism, which in our analysis has been shown to involve the Implicative as well as the Applicative principle. Again the construction of the logical

<sup>1</sup> Cf. Chapter I, p. 11 and onwards.

calculus involves the Implicative as well as the Applicative principle, and will be discussed later. Before proceeding to this topic, we must complete our account of mathematical demonstration by an analysis of *mathematical induction*, which also involves both principles.

§ 6. Mathematical induction assumes a unique place in logical theory. It resembles other forms of demonstrative induction, which will be discussed in a later chapter, where it will be shown that the universal mark of this type of induction is that the conclusion demonstratively inferred asserts for every case what has been asserted in one premiss for a single case. The possibility of such demonstration rests upon the logical character of *the other* premiss, which may be of different types, each type yielding a different form of demonstration. The distinctive characteristic of mathematical induction is that it is concerned with finite integers. These constitute a discrete series beginning with the integer 1, and proceeding step by step in the construction of successive integers. The generation of each successive integer from the preceding is indicated by the operation *plus 1*. Thus, using the illustrative symbol  $n$  to stand for any finite integer, the operation symbolised as  $n + 1$  will yield the next following integer. This construction defines the general conception of a finite integer which is fundamental for arithmetic. The method of mathematical induction introduces the notion of function. Thus  $f(n)$  will be used to stand for any proposition<sup>1</sup>

<sup>1</sup> The functions previously adduced were *mathematical*, i.e. constructs yielding quantities, whereas the function here introduced is *propositional*, i.e. a construct yielding a proposition. And, in general, the equating of two mathematical functions yields a propositional function.



about the specific integer  $n$ , where variation of form will be represented by changing  $f$  into  $\phi$  say, and variation of reference by changing  $n$  into  $m$  say. The argument in its general form will consist of the following assertions of two premisses and of the inferred conclusion:

*Implicative Premiss:* 'The proposition  $f(n)$  would imply the proposition  $f(n+1)$ ' for every finite integer  $n$ .

*Categorical Premiss:* ' $f(1)$ ' holds.

*Conclusion:* Therefore ' $f(n)$ ' holds for every finite integer  $n$ .

In this argument we observe that the conclusion states categorically what is stated hypothetically in the implicative premiss; and further that it predicates for every case what is predicated for a single case in the categorical premiss. Its demonstrative force may be shown by resolving the argument into a succession of steps. Thus, by the applicative principle, we may replace in the implicative premiss  $n$  by 1, and this yields the assertion ' $f(1)$  would imply  $f(2)$ '; then, adding the categorical premiss ' $f(1)$ ,' we infer, by the implicative principle, ' $f(2)$ .' Again, replacing  $n$  by 2, ' $f(2)$  would imply  $f(3)$ ,' and, adding to this the conclusion of the preceding inference, we may infer ' $f(3)$ .' If this process is indefinitely continued we are enabled, by use merely of the applicative and implicative principles, to infer successively  $f(2)$ ,  $f(3)$ ,  $f(4)$ , etc., for every finite integer. The whole argument therefore rests merely upon the same principles as are involved in ordinary deduction; and yet the inference is of the nature of induction, because the conclusion is a generalisation of the same

formula that the categorical premiss lays down only for a single case.

The following is a simple application of mathematical induction:

Let  $f(n)$  stand for the proposition: 'The sum of the first  $n$  odd integers  $= n^2$ .' We have first to establish the implicative premiss, viz.,

$$'f(n) \text{ would imply } f(n+1).'$$

Now  $f(n)$  is the proposition

$$'1 + 3 + 5 + 7 + \dots + (2n-1) = n^2,'$$

and  $f(n+1)$  is the proposition

$$'1 + 3 + 5 + 7 + \dots + (2n-1) + (2n+1) = (n+1)^2.'$$

Here the left hand side of the equation  $f(n+1)$  is obtained from that of  $f(n)$  by adding  $(2n+1)$ .

Hence, by the formula for the square of the sum of two numbers: viz.,

$$(n+1)^2 = n^2 + (2n+1),$$

the conclusion is established that

$$'if f(n) \text{ holds, then } f(n+1) \text{ would hold}.'$$

Now  $f(1)$  holds; for  $1 = 1^2$ . (Also  $f(2)$  holds; for  $1 + 3 = 2^2$ ; and  $f(3)$  holds; for  $1 + 3 + 5 = 3^2$ .)

Hence, having established the implicative premiss ' $f(n)$  would imply  $f(n+1)$ ,' and the single categorical premiss ' $f(1)$ ,' the required universal ' $f(n)$ ' has been proved.

§ 7. In this account of the principles employed in establishing general algebraical formulae, special emphasis has been laid on the novelty of the conclusion as compared with the familiarity and obviousness of the



premisses (including the axioms) from which the conclusion is drawn. This summary account of the methods and results of deductive reasoning enables us to meet what has been called the paradox of inference in a more direct way than that explained in Chapter I. For the existence of the mathematical calculus, where the conclusions are absolutely unknown to those who start by admitting as self-evident the fundamental premisses, constitutes a direct refutation of the arbitrary dictum that for valid inference the conclusion must not contain more than what is already known in asserting the premisses.

The notion of a calculus is generally associated with elaborate symbolism, which renders possible the more complex deductive processes in logic and mathematics. As a question of history, there is no doubt that the introduction of such simple symbols as  $+$ ,  $-$ ,  $\times$ , created a revolution in mathematical science, and rendered it possible to make advances otherwise unattainable. Again it is an equally noteworthy historical fact that the best formal logicians, such as Leibniz and Lambert, were comparatively unsuccessful in their attempt to develop a logical calculus, which was first started by Boole on lines followed by all subsequent symbolists who advanced the science. Boole's method was simply to import the familiar symbols of elementary arithmetic into logic, making use of the fundamental formulae with which algebraists were already conversant. In this way he created the first great revolution in the study of formal logic, and one that is comparable in importance with that of the algebraical symbolists in the sixteenth century. I think, however, that Boole's procedure has

led to considerable confusion with regard to the relations between the logical and the algebraical calculus, inasmuch as he seems to have supposed—in common with many logicians of his time—that the advance achieved by introducing mathematical formulae into logic made logic into a department of mathematics. This attitude of Boole's obstructed, for a considerable period, the investigation of the foundations of mathematics, which demanded the reversal of the relationship between the two sciences. It is under the influence mainly of Peano and of the new mathematicians such as Cantor, that we now recognise mathematics to be a department of logic. The current phrase *mathematical logic* is ambiguous inasmuch as it may be understood to mean either the logic of mathematics or the mathematics of logic. Now, in my view, the logic or rather philosophy of mathematics is a study which ought to dispense entirely with symbolic language. It must, of course, explain the nature of symbols and of symbolic methods, and account for the extraordinary power of symbolism in deducing with absolute security previously unknown formulae. But the philosophical exposition of the deductive power of mathematics must be treated in language the understanding of which requires thought of a profounder nature than that required in merely following symbolic rules. As indicated in the chapter on Symbolism and Functions, the essential purpose of symbolism is to economise the exercise of thought; and thus symbolic methods are worse than useless in studying the philosophy of symbolism or of mathematics in particular. The phrase 'mathematics of logic,' on the other hand, merely indicates a certain line of development of logic, in which

deductive processes are reduced to strictly demonstrative form by means of a symbolism founded on explicitly logical axioms. The important advances in this direction have been systematised with extraordinary success in Whitehead and Russell's great work *Principia Mathematica*, where it is shown how pure mathematics can be actually developed from pure logic. The value of the work consists, therefore, in reducing mathematics to logic, and not at all in reducing logic to mathematics. I shall attempt hardly any criticism of their formal development of the science, and shall here confine myself to the principles which enter into its very elementary foundations.

§ 8. In contrasting the mathematical developments of logic with the ultimate foundations of the science, it will be convenient to use the terms premathematical and mathematical logic, the latter of which introduces certain novel conceptions, strictly formal in character, in addition to those employed in the former. There are certain notions common to the premathematical and mathematical departments of logic, and of these we have already discussed the nature of functions, illustrative and short-hand symbols, variables, brackets, etc., which before Peano and Russell had not received adequate recognition in logical teaching; they apply, however, over a wider field than mere mathematics, and must therefore be transferred without modification from the narrower science back to logic. The term 'formal' as applied to these conceptions means that they are to be understood by the logician as such, and they include, besides those primitive ideas which are to be understood without definition, also derivative ideas which are com-

pletely defined in terms of primitive ideas. For example, the notions of implication, alternation, disjunction and negation are formal, and of these we may take negation and alternation as understood without definition, while the others can be defined in terms of these two<sup>1</sup>. Again logical categories and sub-categories such as substantives proper, primary and secondary adjectives and propositions, come under the head of formal conceptions. There are also specific adjectives and relations, such as true, probable, characterised by, comprised in, identical with, which are formal; and, though some of them are ultimately undefinable, the understanding of all of them is essential to logical analysis. In contrast to these formal adjectives, such adjectives as *red*, *hard*, *popular*, *virtuous*, etc. are termed material, because their meaning is unessential to the explication of logical forms. In premathematical logic formulae are established for all adjectives as such, or for a limited set of adjectives comprised in such a sub-category as that of secondary adjective. The range over which these formulae hold must be said to be *material*, though it necessarily comprises adjectives which may happen to be formal, i.e. to have specifically logical significance. Passing from premathematical to mathematical logic, we find that *new* specific adjectives, having essentially logical significance and coming therefore under the head of formal conceptions, are introduced. We may specially mention *integers* and *ratios*. Integer is a logical sub-category

<sup>1</sup> This, at any rate, is the procedure of the *Principia Mathematica*; but, while undoubtedly permissible from the point of view of the logical *calculus*, it is open to serious philosophical criticism, which I have given elsewhere.

comprised in the general category *adjective*; and ratio is a logical sub-category comprised in the general category *relation*; but what constitutes the new feature in mathematical logic is that each specific integer and each specific ratio has itself essentially logical significance, while at the same time formulae hold for all integers and again for all ratios. Premathematical logic on the other hand can only establish formulae holding for adjectives in general or for secondary adjectives in general. This distinction carries with it the further result that premathematical logic can only use illustrative adjectival symbols as variables over a range of variation covering the whole category *adjective*, or the whole sub-category *secondary adjective*; while in mathematical logic there occur illustrative symbols for variables covering the range, in the one case *integer*, in the other *ratio*. Consider for example such an illustrative symbol as *m* in ordinary or premathematical logic. The specific values that can be substituted for this variable are material; for the formal character of such of them as have specifically logical significance is irrelevant to the truth of the formulae. In mathematical logic, on the other hand, *all* the specific values which can be substituted for a symbol *m* standing for any integer, say, or a symbol *t* standing for any ratio, denote formal conceptions. Again it is obvious that, besides the formulae which hold for adjectives in general, there are innumerable additional formulae holding for integers or for ratios; and this accounts for the variety and complexity of mathematics as compared with premathematical logic. But the essential distinction between the two sciences—or rather the two departments of logical science—lies in the point

already urged, namely that every specific adjective within a certain mathematical range has itself a logically determined value; whereas no logically determined value can be assigned to adjectives in general which enter into premathematical logic. This distinction may be summed up in other words by taking the two antitheses material and formal, and constant and variable, which combined give the four cases formal variables, formal constants, material variables and material constants. Now premathematical logic uses formal constants and material variables (and also in Mr Russell's work material constants), *but nowhere formal variables*. On the other hand mathematics uses formal constants, material variables, *and also formal variables*. It is therefore the use of formal variables that fundamentally distinguishes mathematics from premathematical logic.

§ 9. To continue our account of the relation between the premathematical and mathematical departments of logic, we must next define and illustrate the nature of those formal elements which are never expressed by variable symbols, and therefore come under the head of formal constants. To these, the name *connectives* will be given. The first division under this head includes what are known as operators in mathematics, such as *plus*, *minus*, *multiplied by*, *divided by*, as well as analogous logical operators such as *and*, *or*, *not*, *if*. Thus the operation '*m + n*,' where *m*, *n* stand for determinate numbers, yields a certain determinate number; and analogously the operation '*p and q*,' where *p*, *q* stand for determinate adjectives, yields a certain determinate adjective. This is most clearly seen when a proper name has been invented to stand for the compound

construct as well as for each of the constituents themselves: thus, the operation 'three-plus-five' yields the number 'eight'; the operation 'rational-and-animated' yields the adjective 'human.' The analogy goes one step further when, in place of the simple predication *yields*, we use the complex yields-what-is-yielded-by: thus, the operation '*m* plus *n*' yields-what-is-yielded-by the operation '*n* plus *m*'; the operation '*p* and *q*' yields-what-is-yielded-by the operation '*q* and *p*.' Now neither in logic nor in mathematics is it ever required to use illustrative or variable symbols to stand for formal operators like *plus* or *and*—the reason being that no formula which holds for one operator will hold if we substitute indiscriminately any other operator. Hence, if symbols are used for formal operators, these come under the head of short-hand symbols, and never under the head of illustrative or variable symbols. Thus the operators both of logic and of mathematics enter as formal constants, never as variables.

In the second division of connectives are to be included certain relational predications which must be systematically illustrated and classified according to their different properties. Of these, the five of most fundamental importance are the relational predications: identical with, implied by, characterised by, comprised in, included in, together with their cognates. These are formal, and to represent them I shall introduce the short-hand symbols:  $\hat{=}$ ,  $\hat{\imath}$ ;  $\hat{\lambda}$ ,  $\hat{\lambda}$ ;  $\hat{\chi}$ ,  $\hat{\chi}$ ;  $\hat{\kappa}$ ,  $\hat{\kappa}$ ;  $\hat{\nu}$ ,  $\hat{\nu}$  respectively. These five formal connectives are absolutely distinct from one another, although they have been frequently confused by logicians; and this distinctiveness is sufficient to account for the fact that they are

never represented by variable symbols for which one could replace another. Thus: in the predication  $x\hat{=}\hat{y}$ , the symbols  $x$  and  $y$  must stand for entities belonging to one and the same assigned category; but, in the predication  $x\hat{\kappa}\hat{y}$ ,  $x$  and  $y$  must stand respectively for an item or member and an enumeration or class; and, in the predication  $x\hat{\chi}\hat{y}$ ,  $x$  and  $y$  must stand respectively for a substantive and an adjective. Again, while  $\hat{\imath}$  connects entities belonging to *any* the same category,  $\hat{\lambda}$  connects only propositions or adjectives or relations; and  $\hat{\nu}$  connects only classes or enumerations of the same order. And yet again: the relation *identity* is reflexive, symmetrical, and transitive; but the relations *characterised by* and *comprised in* are a-reflexive, a-symmetrical, a-transitive; while the relations *implying* and *included in* are reflexive and transitive but neither symmetrical nor a-symmetrical. The five connectives above enumerated may be said to be on the borderland between premathematical and mathematical logic. There are, however, many formal connectives which belong exclusively to mathematics, of which the most fundamental is *equals* universally represented by the short-hand symbol  $=$ . There is serious danger of confusing *equal-to* with *identical-with* because they agree in possessing the properties reflexive, symmetrical and transitive (to the consideration of which we shall have to return later). Other important connectives in logic and algebra are derivative from those above enumerated as fundamental. Classifying fundamentals and derivatives according to their properties we have the following table, where the initials  $F$ ,  $S$ ,  $T$  stand respectively for reflexive, symmetrical and transitive, and the suffix  $a$

means 'for all cases,' *e* 'for no cases,' and *oi* 'for some but not all cases.'

| Formal Relations                                                             | Properties          |
|------------------------------------------------------------------------------|---------------------|
| Identical, equal, co-implicant, coincident ... ..                            | $F_a S_a T_a$       |
| Differing by unity, co-opponent, co-remainder ...                            | $F_e S_a T_e$       |
| Greater, less, sub- and super-implicant, sub- and super-incident ... ..      | $F_e S_e T_a$       |
| Sub- and super-opponent, sub- and super-remainder ...                        | $F_e S_e T_{oi}$    |
| Not greater, not less, implying, implied by, included in, including ... ..   | $F_a S_{oi} T_a$    |
| Other than, unequal, disjunct, alternant, co-exclusive, co-exhaustive ... .. | $F_e S_a T_{oi}$    |
| Not disjunct, not alternant, not co-exclusive, not co-exhaustive... ..       | $F_a S_a T_{oi}$    |
| Not implying, not implied by, not included in, not including ... ..          | $F_e S_{oi} T_{oi}$ |
| Comprising, comprised in, characterising, characterised by ... ..            | $F_e S_e T_e$       |

§ 10. Not only formal constants but also material variables enter in the same way into mathematics as into premathematical logic. The particular values which any material variable may assume are unessential for pure logic and pure mathematics; and enter as significant factors only into applied logic or applied mathematics. For example, a variable representing *any* substantive or *any* adjective is replaced by a particular substantive or a particular adjective only when the general formulae established by logic are applied to concrete propositions. Similarly the purely formal notion of magnitude or of quantity, which enters into mathematics, is applied to several different species and sub-species such as mass, volume, intensities of different kinds, etc., the differentiae of which, not being expressible in terms of pure mathematical conceptions, must be determined materially. Thus, for instance, in the mathematical

formula  $3q + 5q = 8q$ ,  $q$  enters as a material variable standing for any quantity; and 3, 5, 8, =, +, as also the category *quantity* itself, enter as formal constants. But in applying the material variable  $q$  to deduce the equation

$$3 \text{ feet} + 5 \text{ feet} = 8 \text{ feet, or } 3 \text{ ohms} + 5 \text{ ohms} = 8 \text{ ohms,}$$

the terms foot, ohm, as species of the genus quantity, have to be defined by means of conceptions outside the range of pure mathematics. In this way we see that variable symbols—material as regards their range of application—entering into premathematical and mathematical logic, assume their particular values when logical theorems are applied to experimental matter. Having shown then, as regards both formal constants and material variables, that general logic agrees in all respects with mathematics, the conclusion follows that the latter fundamentally differs from the former in the sole fact that it introduces formal variables.

§ 11. Before examining the characteristics of the specifically mathematical notion 'equals' upon which its symmetry and transitiveness depend, we will consider the wider problem of relations in general possessing these two properties. There is one mode of constructing such relations which has very wide application and is of great importance in logical theory, viz.

' $x$  is  $\hat{r}$  to *the* thing that is  $\check{r}$  to  $z$ .'

Here the word *the* indicates that  $\hat{r}$  is a many-one relation. I shall call 'the thing' to which reference is made in the above formula the *intermediary* term, and the relation  $\hat{r}$  the *generating* relation. Thus, given an intermediary term and a many-one generating relation, we can always construct by (what is called) *relative*



*multiplication* a derived relation which is symmetrical and transitive. Representing the intermediary by the symbol  $y$ , the relation of  $x$  to  $z$  may be otherwise expressed by the conjunctive proposition:

' $x$  is  $\hat{r}$  to  $y$  and  $z$  is  $\hat{r}$  to  $y$ ,'

where it is to be understood that there *is* some uniquely determined entity (say  $y$ ) to which  $x$  and  $z$  stand in the relation  $\hat{r}$ ; i.e.  $\hat{r}$  is a many-one relation.

Now the theorem that any relation so constructed is symmetrical and transitive requires no discussion and is universally admitted; but the converse theorem—that any symmetrical and transitive relation can be exhibited by this mode of construction—cannot be assumed to be true without careful examination. To this converse theorem Mr Russell gives the name 'the principle of abstraction'; and professes to have proved its truth by a process involving highly complicated symbolism. It is quite easy, however, to explain the nature of his proof without recourse to such symbolism. Thus, let  $\hat{r}$  be a symmetrical and transitive relation; then, in order to prove the theorem, we have to discover an intermediary entity and a generating relation in terms of which  $\hat{r}$  may be constructed. The intermediary entity for the relational predication ' $x$  is  $\hat{r}$  to  $z$ ' is, in Mr Russell's proof, 'the class of things comprising  $x$  together with everything such as  $z$  for which " $x$  is  $\hat{r}$  to  $z$ " holds.' The required generating relation  $\hat{r}$  is the relation of *being comprised in*; hence the proposition ' $x$  is  $\hat{r}$  to  $z$ ' is resolved into the form:

$x$  is comprised in the class  
(defined as comprising everything to which  $x$  is  $\hat{r}$ )  
which comprises  $z$ .

Here the intermediary entity is a class uniquely defined in terms of  $x$  and  $\hat{r}$ , and therefore the relation in which  $x$  or any other item stands to the intermediary is a many-one relation. Now what Mr Russell has succeeded in proving in this way is proved with absolutely demonstrative validity; but my first comment is: has he proved what he undertook to prove? In one sense he has proved too much, and in another sense he has proved nothing whatever that is relevant. He has proved too much in the sense that he has discovered an intermediary entity which would, *mutatis mutandis*, apply to every possible symmetrical and transitive relation, such as contemporaneous, compatriot, co-implicant, co-incident, as well as equal. Thus he has proved that, for the resolution of the relation *equals*, we must take as intermediary 'the class of quantities equal to any given quantity'; for the relation *contemporaneous*, 'the class of events contemporaneous with any given event'; for the relation *compatriot*, 'the class of persons that are compatriots of any given person'; and so on. But what he set out to discover as the required intermediary was, in the case of equality, a *certain magnitude*; in the case of compatriot, a *certain country*; in the case of contemporaneous, a *certain date*; and so on. He has *not* proved that there is a certain magnitude that all equal quantities possess; nor a certain country to which all compatriots belong; nor a certain date to which all contemporaneous events are to be referred. Moreover, in taking as his intermediary a certain uniquely determined *class*, it seems obvious that Mr Russell's alleged proof is incomplete, unless we can assert that there *are* such entities as classes, and the



validity of this assertion is explicitly denied by him : or rather he holds that there is no necessity in the deductions of logic and mathematics to assume that there are classes, although without this assumption his proof of the principle of abstraction completely breaks down.

I do not, however, wish to press my criticism of Mr Russell further, but rather to expound what appears to me to be the true view on the nature of abstraction. The cases in which the principle comes into consideration may be distinguished according as the intermediary is of the nature of a substantive such as *country*, or of the nature of an adjective such as *magnitude*. In applying the attempted proof of the principle of abstraction to such a relation as *compatriot*, Mr Russell argues as if we knew this relation to be symmetrical and transitive independently of our knowledge that a person can belong-to ( $\hat{r}$ ) only one country ( $y$ ); whereas it is obvious that we have constructed the derivative relation *compatriot* by means of the prior notions *country* and *belonging-to*. Hence, no such case as *compatriot* can be used to prove the principle of abstraction, but only to illustrate the theorem of which the principle of abstraction is the converse. Where the intermediary is adjectival, e.g. colour, pitch, magnitude, the principle directly raises the issue of the connection and distinction between a determining adjective and the class that it determines. In the case of an adjectival intermediary, our general formula

‘ $x$  is  $\hat{r}$  to the term (say  $y$ ) that is  $\hat{r}$  to  $z$ ’

must be expressed in a special form in which the generating relation ( $\hat{r}$ ) is to stand for characterised-by ( $\hat{\chi}$ ),

and the intermediary term ( $y$ ) is to stand for a *specific determinate under a specific determinable*, thus:

‘ $x$  is characterised-by the determinate adjective that characterises  $z$ .’

Here the uniqueness of the intermediary term is secured by the disjunctive principle of adjectival determination expressed (Part I, Chapter XIV) in the form: ‘Nothing can be characterised by more than one determinate under any assigned determinable.’ Now, since any one substantive may be characterised under many different determinables, the intermediary term must specify the determinable, or ground of comparison, upon which the symmetry and transitivity of the derived relation depend. Thus,

‘ $x$  is characterised by the colour that characterises  $z$ ,’  
or ‘ $x$  is characterised by the shape that characterises  $z$ ,’  
or ‘ $x$  is characterised by the size that characterises  $z$ .’

Any of these three propositions may be significantly asserted of the same subjects  $x$  and  $z$ , if these are bounded surfaces distinguished from one another by determinate localisation; and the relation of  $x$  to  $z$  thus constructed is transitive (as well as symmetrical) provided that the colour, shape or size is strictly determinate. With this proviso, we may say that  $x$  and  $z$  are equivalently coloured, equivalently shaped or equivalently sized, as the case may be. Such symmetrical and transitive relations between the *substantives*  $x$  and  $z$  must be distinguished from the symmetrical and transitive relation *identity* which holds between the *adjectives* described as the colour of  $x$  and the colour of  $z$ , the shape of  $x$  and the shape of  $z$ , or the size of  $x$  and the size of  $z$ .

Now *magnitude*—like any other adjectival determinable—must first be abstracted as a character in order that by its means we can construct the class of equally sized objects. Thus it is just as absurd to define the size of  $x$  in terms of 'the class of objects that are equal in size to  $x$ ' as to define the colour of  $x$  in terms of 'the class of objects that are equivalent in colour to  $x$ .'

§ 12. To secure that the relations constructed by means of the above formula shall be symmetrical and transitive, it is necessary to specify, not only such differences as those between colour, shape, etc., but also differences within the general notion *magnitude*, constituting various kinds or species of magnitude. For just as colours and sounds are incomparable with one another, since they must be characterised under different determinables, so there are distinct determinables subsumable under the superdeterminable *magnitude*. Taking some of Mr Russell's suggestive examples, we note that the magnitude of pleasure predicable of an experience is incomparable with the magnitude of area predicable of a surface, and that these again are incomparable with the magnitude of duration predicable of an event. Hence pleasure-magnitude, area-magnitude, duration-magnitude, are three distinct determinables, predicable only of experiences, surfaces, and events respectively. In ordinary usage the word *magnitude* is omitted when reference is made to the determinables in question; but in specifying the 'area' of a surface, we are in point of fact specifying a kind of magnitude; so in specifying the 'duration' of an event we are specifying another kind of magnitude; and in specifying the 'pleasure' of an experience, we are specifying yet another kind of magnitude. The

analogy here drawn between area or duration on the one hand, and pleasure on the other will probably be disputed because pleasure is so often used in its concrete sense to mean 'pleasurable experience' as well as in its abstract sense to mean 'the pleasure of a (pleasurable) experience.' Now it happens that a pleasurable experience may be characterised under at least two different determinables of magnitude; viz. pleasure-magnitude and duration-magnitude, the latter of which applies in the same sense to any event whatever that may last through a period of time. Here it is important to note that pleasure-magnitude and duration-magnitude, etc. are not determinates under the one determinable *magnitude*, but different species included in the genus magnitude. They may therefore be conveniently termed sub-determinables of magnitude, each generating its own determinates, which are incomparable with the determinates generated by any other. Thus magnitude does not generate its sub-determinables in the way in which a determinable generates its determinates. An experience, a surface, an event are substantives belonging to different categories of which pleasure, area, or duration may be respectively predicated as adjectives; but a *specific* pleasure-magnitude, or area-magnitude, or duration-magnitude is related to its respective species of magnitude as a determinate to its determinable. We shall proceed in the next chapter to examine and classify the fundamental kinds of magnitude, to which reference is here made.

§ 13. It remains to point out one highly important characteristic which distinguishes pure or pre-mathematical logic from mathematics proper. In both branches, the two principles of inference termed *Applicative* and

*Implicative* are employed in the procedure of functional inference, and these alone. But the peculiarity of pre-mathematical deduction is that it lays down two *formulae of implication* (either as primitive or as derived) which are virtually equivalent respectively to the Applicative and Implicative Principles themselves. The formulae in question may be thus expressed:

(1) *Applicative formula*: Any predication that holds for every case  $x$  would formally imply that the same predication holds for a given case  $a$ .

(2) *Implicative formula*: For any case  $x, y$ , the compound " $x$ " and " $x$  would imply  $y$ " would formally imply ' $y$ '.

We must, therefore, explain the distinction between *Principles of Inference*, on the one hand, and *Formulae of Implication*, on the other hand. In all formulae of implication, the implicans and implicate stand indifferently for propositions that are to be *materially* or *formally* certified. But, when a formula of implication is used as a premiss in the process of deduction, its implicans must first be formally certified in order that its implicate may be formally certified. This inference is made by a direct *application* of the *implicative* principle. And again, every formula of implication holds for *all* cases coming under an assigned form; hence the inferences from any formula of implication are made by a direct *application* of the *applicative* principle. The fact that every step by which we advance in the building up of the logical calculus requires both the Applicative and the Implicative principles of inference, *and these alone*, establishes their sovereignty over all deductive processes.

## CHAPTER VII

### THE DIFFERENT KINDS OF MAGNITUDE

§ 1. THE term *magnitude*, as is suggested by its etymology, denotes anything of which the relations greater or less can be predicated; and it is only if  $M$  and  $N$  (say) are magnitudes of the same kind that  $M$  can be said to be greater or less than  $N$ . I have taken *magnitude* to be an adjectival determinable, or rather a class of adjectival determinables including several distinct kinds. That of which a determinate magnitude of a specific kind may be predicated stands, relatively to its magnitude, as substantive to adjective; but it may be either an existent, i.e. substantive proper (in which case the magnitude predicated is a primary adjective) or itself an adjective (in which case the magnitude predicated is a secondary adjective). In order to keep clear the distinction between the adjectives of magnitude themselves and the substantives of which magnitude is predicable, a separate terminology ought strictly to be applied to the latter. A striking case where language supplies us with the logically required terminological distinction is that of 'longer' and 'shorter' predicated of *lines*—to the *lengths* of which the terms 'greater' and 'less' are applied. It would be convenient, for the purposes of a general exposition of magnitude, to restrict the application of the terms 'greater' and 'less' to magnitudes, and to adopt the corresponding terms 'larger' and 'smaller' for that of which the

magnitudes are predicated. For example: the class *compositae* is larger or smaller than the class *violaceae*, according as the number of *compositae* is greater or less than the number of *violaceae*<sup>1</sup>; the period 1815 to 1832 may be called larger than the period 1714 to 1720, inasmuch as the temporal magnitude of the former is greater than that of the latter. Now for every distinct kind of magnitude there is a corresponding distinct kind or category of entity of which it can be predicated; and hence, though it is strictly illogical, yet it is legitimate and usual to apply the same terms, such as extensive and intensive, to distinguish both between the different kinds of magnitude and between the corresponding different kinds of entities which bear to the magnitude the relation of substantive to adjective. From these preliminary remarks, we may pass to an examination of the nature of different kinds of magnitude, beginning with *number*, which is the most fundamental of all.

§ 2. Integral number is an adjective exclusively predicable of what we call classes, including enumerations; two classes being said to be numerically equal when the number predicable of the one is identical with that predicable of the other. I think it is legitimate to maintain that the two notions class and number are not independently definable, but each definable only in its relation, the one as the only appropriate substantive for

<sup>1</sup> This may mean either that the number of *existing plants comprised* in the genus is greater or less, or that the number of *infimae species included* in the genus is greater or less. It is obvious that these two modes of determining numerical comparison do not necessarily tally. It will be shown later that the same distinction holds as regards the number of *points* in a line and the number of linear *parts* (equal or unequal) into which it may be exhaustively and exclusively divided.

the other as its only appropriate adjective. The common habit of representing classes by closed figures may lead to the false supposition that the members of a class can as such be arranged in some kind of proximity to one another within an enclosed space. But when the items to be comprised in a class have relations metaphorically called near or far, they constitute not merely a class but a series or ordered class. Now in modern mathematics the appropriate number-adjective of a class conceived independently of any arrangement or order of its items, is known as a cardinal number; whereas of a series or ordered aggregate the appropriate numerical adjective is known as an ordinal number. When a class or enumeration comprises a finite number of items, then, in whatever order the items may be enumerated, we reach the same ordinal number, and this number agrees with the cardinal number; but for transfinite aggregates, which have been introduced into modern arithmetic, this agreement no longer holds; and consequently the fundamental distinction between ordinal and cardinal numbers is required. Readers are referred particularly to Mr Russell's *Principles of Mathematics* for the full development of this topic, which is outside the compass of my work.

§ 3. The psychological aspect of number is revealed by analysing the process of counting. In this process we establish numerical equality between a set of *things*, on the one hand, and a set of *number-names* temporarily attached to the things, on the other hand. Hence counting is a special, and, in some respects, a unique case of correlation between the things upon which names are imposed and the names that are imposed upon the

things. Ideally language requires that any given proper name should denominate one and only one thing, and conversely that any given thing should be denominated by one and only one proper name; or briefly, that there should be a one-one correlation between the names of things and the things named. If this relation held, it would follow that the class of names would be numerically equal to the class of things named. Actually, however, this ideal is not realised; for the same thing often has many names, and the same name is often attached to many things. It is worth pointing out that there may still be numerical equality in spite of there not being a one-one correlation between names and things named. For example: let  $R, Q, M, T, U$  be a set of names, and  $\kappa, \sigma, \theta, \chi, \phi$  a set of things named. Then suppose that

$R$  names  $\kappa$  or  $\sigma$ ;  $Q$  names  $\kappa$  or  $\sigma$ ;  $M$  names  $\sigma$  or  $\theta$  or  $\chi$ ;  $T$  names  $\chi$ ; and  $U$  names  $\theta$  or  $\phi$  or  $\kappa$  or  $\chi$ ; so that

$\kappa$  is named  $R$  or  $Q$  or  $U$ ;  $\sigma$  is named  $R$  or  $Q$  or  $M$ ;  $\theta$  is named  $M$  or  $U$ ;  $\chi$  is named  $M$  or  $T$  or  $U$ ; and  $\phi$  is named  $U$ .

Here the denominating correlation is not one-one but many-many, and yet the names and the things happen to be numerically equal. How then do we establish the fact that the number of items in the enumeration  $R, Q, M, T, U$  is the same as that in the enumeration  $\kappa, \sigma, \theta, \chi, \phi$ ? What we do, where there is no *factual* correlation, is to institute what I shall call a *factitious* correlation; by which I mean one which is not inherent or objective, but arbitrarily imposed by the counter. In the adduced instance—in order to establish the numerical

equality between the enumerations  $R, Q, M, T, U$ , and  $\kappa, \sigma, \theta, \chi, \phi$ —we must mentally attach, either in thought or in figurative imagery,  $R$  say to  $\theta$ ,  $Q$  to  $\phi$ ,  $T$  to  $\sigma$ ,  $U$  to  $\kappa$ ,  $M$  to  $\chi$ ; where the items of the two sets have been indiscriminately permuted and attached. We can now analyse the mental act of counting as a special case of factitious correlation. The essential psychological requisite is that we should *learn to enumerate a set of arbitrary names in a fixed or invariable temporal order from the first onwards*; and these names are attached *temporarily* to the objects to be counted, in this respect differing from names in general which have *fixed* denotation. For example: let us arrange the names  $U, R, Q, M, T$  in the following order:  $M, Q, R, T, U$ ; and *temporarily* attach these names as follows:  $M$  to  $\chi$ ,  $Q$  to  $\phi$ ,  $R$  to  $\theta$ ,  $T$  to  $\sigma$ ,  $U$  to  $\kappa$ . Thus the set of names have to be attached in a *fixed* order, one by one, to the set of things taken in *any* order. What is logically required to avoid mistake is that the enumeration of the things should be both exhaustive and non-repetitive—a condition which children and savages often find difficult to fulfil. Now, inasmuch as the number-names  $M, Q, R, T, U$  are always attached in an invariable order, the *last* number named indicates unequivocally the number of the counted set of objects. In other words, the cardinal number of any enumerable set of objects is unambiguously indicated by the ordinal number of the correlated number-names. Historically the letters of the alphabet, having been memorised in a fixed order, served also as the written symbols for numbers; but their employment for this purpose could not be extended to *all* numbers, since an alphabet



necessarily consists of a limited number of letters. Moreover it is psychologically impossible to memorise an endless list of names. Hence it was necessary to invent some system which would render it possible to count any set of things, however large. The Roman, Greek and Hebrew alphabets were employed for this purpose with more or less success, but were finally superseded by the Arabic notation in which *place-value* was given to the symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, and the symbol 0 was added. These ten symbols serve as *proper* names of numbers, all other numbers being expressed by names constructed out of these. Thus the compound word twenty-four or the compound symbol 24 is analysable as meaning 'two tens plus four,' and therefore to be understood in terms of the operations of multiplication and addition. Such compound symbols or words are not proper names of numbers like two, ten, or four, but may be called constructed names. The elementary learner of arithmetic must, in fact, reverse the logical order of thought, and understand the processes of multiplication and addition before he can intelligently learn to count beyond twenty or so, or understand what is known as the decimal system of notation.

§ 4. We now pass from the psychological analysis of counting to the consideration of its underlying logical principles. Counting is a special case of one-one correlation, the peculiar characteristics of which are (1) that a prescribed set of name-items have to be memorised in a definite serial order; and (2) that the correlations are factitious. As regards (1) the mental process of counting, which involves order, must be contrasted with

one-one correlations in general which are irrespective of order. As regards (2) factitious correlations must be contrasted with such correlations as husband and wife, denominating and denominated by (in an ideal language) etc., in which any given husband is correlated with a determinate wife, or any given proper name with a determinate thing. Now in my view, factitious correlations are essentially necessary in the *general* theory of numerical equality; though they never enter into the abstract deductions of arithmetic. On the necessity of factitious correlations, recognised authorities, above all Mr Russell, are opposed to me. Their definition of the numerical equality of two sets of things is, in effect, formulated as follows: 'There is a one-one relation of any member of the one set to some member of the other set.' But it seems to me essential to distinguish the statement that 'the items *can be* correlated one to one' from the statement '*there is* a one-one correlation'; the former points to a factitious, the latter to a factual correlation. There need be no relation at all depending on the nature of the items themselves comprised in the two sets, that would determine which item of the one set should be attached to any given item in the other. If relations are treated extensionally, i.e. as mere substantive-couples, then it is of course a matter of fact that two numerically equal classes contain couples of items, one of which is comprised in the one class and the other in the other; but I know of no sense in which the two members of the couple are *related* the one to the other, except that the one is temporarily attached in thought by some thinker to the other. Apart from this *factitious* coupling, there is *no* one-one relation,



subsisting between any given item of the first set and any determinate item of the second, that would not equally subsist between the given item and any *other* item arbitrarily selected from the second set. Thus the establishment of numerical equality between two finite classes requires *in general* factitious correlations. On the other hand, the only mode of establishing numerical equality between *infinite* classes is to discover *factual*, or more specifically *formal*, correlations. The formal correlations required in pure arithmetic, finite and transfinite, are what may be called functional; and, for the purposes of this elementary exposition of the logic of arithmetic, the notion of functional correlation must be introduced and explained.

§ 5. Using the symbol  $\hat{f}$  for any function and  $f$  for its converse, the relation  $n$  to  $\hat{f}(n)$  will be one-one; provided that  $n$  determines uniquely the value of  $\hat{f}(n)$  and  $\hat{f}(m)$  determines uniquely the value of  $f\{\hat{f}(m)\}$ . For example: let  $\hat{f}(n)$  stand for  $n+7$ , then  $f(m)$  will stand for  $m-7$ ; and the integers from 1 to  $n$  (inclusive) can be correlated one to one with the integers from 8 to  $n+7$ ; each integer in the second series being given by adding 7 to the corresponding integer in the first, and each in the first series by subtracting 7 from the corresponding integer in the second. Similarly, if  $\hat{f}(n)$  stands for  $n \times 7$ , then  $f(m)$  will stand for  $m \div 7$ ; and the integers from 1 to  $n$  can be correlated one to one with the multiples of 7 from 7 to  $7n$ . In general: if the relation of  $n$  to  $\hat{f}(n)$  is one-one, then the series of values assumed by  $n$  is numerically equal to the series of values assumed by  $\hat{f}(n)$ .

An important application of this theorem is to the case where the integer  $n$  assumes all possible finite

values, obtained from unity by the successive addition of unity. In this case, the simplest illustration is afforded by taking  $\hat{f}(n)$  to stand for  $2n$ . There is then established a one-one correlation of the successive integers 1, 2, 3, 4 ... with the successive integers 2, 4, 6, 8 .... In other words, the number of finite integers is the same as the number of finite *even* integers; although the former series comprises all the *odd* integers and these are not comprised in the latter. Thus, although the aggregate of even integers is a part proper of or *sub-included in* the aggregate of integers, yet the two aggregates are numerically equal. Now we may *define* an infinite number as the number of any aggregate that includes a part proper numerically equal to itself. Thus the instance above cited is the simplest of the many proofs that establish the theorem that *the number of finite integers is infinite*. If the integers are presented in ascending order of magnitude, the series so conceived has a first but no last term and also is discrete in the sense that each term has one and only one immediate successor. The cardinal number of any aggregate that can be so arranged in a series is called  $\aleph_0$ . This is the smallest of infinite cardinal numbers.

The reader must here be referred to the mathematical exponents of the theory of transfinite cardinals and ordinals for further instruction. The most comprehensive account of this theory will be found in Mr Bertrand Russell's work entitled *Principles of Mathematics*.

§ 6. As number and the magnitudes that are derived solely from number may be called abstract, so those which contain a material factor may be called concrete

magnitudes or quantities. Thus duration-magnitudes, stretch-magnitudes, magnitudes of qualitative difference are quantities, because the entities of which they are predicable are defined and differentiated in terms that are not purely logical. This use of the term *quantity* differs from that expressly enjoined by Mr Russell, who defines a quantity as 'an instance or specification of magnitude.' He then proceeds to identify the relation thus indicated in some cases with that of substantive to adjective, and in others with that of determinate to determinable; whereas, in the common language of mathematics, quantity stands to magnitude in the relation of species to genus, with which my use of the term *quantity* corresponds. With regard to quantities the three differentiae which I hold to be fundamental or primitive are extensive, distensive and intensive. The term distensive magnitude is new, and the reason for placing it intermediately between extensive and intensive is that by some logicians it has been included under extensive and by others under intensive magnitude.

An extensive magnitude may be defined as one which can be predicated only of an entity that can appropriately be called a whole. The notion of whole is correlative to the notion of part; and, more precisely, a whole is to be conceived as having parts which can be specifically identified and distinguished independently of their relations of equality or inequality; e.g. a finite line is a whole of the simplest possible kind, under the figure of which all one-dimensional wholes may be metaphorically pictured. Thus a line *CEG* is represented as having the parts *CE* and *EG*, each of which is definitely identifiable for itself and distinguish-

able from the other. The several parts of a whole are of the same nature as the whole, and therefore the construction of a whole out of parts or the division of a whole into parts may always be called homogeneous<sup>1</sup>. The term extensive magnitude has, in fact, been popularly restricted to spatial and temporal wholes; but I shall follow Mr Russell in applying this term also to certain qualitative wholes, e.g. to a continuous aggregate of hues or of pitches. Thus we speak of a scale of hue and a scale of pitch in the sense of a class comprising all specific hues or pitches which are qualitatively intermediate between two terminal hues or pitches. Now the class comprising such determinate items constitutes what is now called a *stretch*; thus a qualitative stretch of hue or of pitch is formally analogous to the period comprising all determinate instants between one instant and another or to the geometrical line comprising all points intermediate between one point and another.

§ 7. It might appear, since the instants comprised in a period and the points comprised in a line are substantial, while the hues or pitches comprised in a qualitative stretch are adjectival, that there is some fundamental logical distinction between these two kinds of stretches. Thus: though either may be metaphorically represented by a line *CEG*, yet, if the points *C*, *E*, *G* stand *actually* for points or instants—these being substantial—the stretch represented is substantial; whereas if *C*, *E*, *G* represent three pitches—pitches

<sup>1</sup> The term *whole* is frequently applied to a construct constituted of *heterogeneous* elements, e.g. to a proposition; but for such a construct the term *unity* is preferable, unity being the genus of which whole is a species.

being characteristics of sound and therefore adjectival—the stretch itself is adjectival. It is open to question, however, whether points or instants are of a substantival nature; and this has been a matter of frequent philosophical dispute. If we regard time and space as existents, then the events which occur at a given date or occupy a given period, like the ink-spots which may be placed at different points or the ink-lines which may be drawn on paper, have as substantives a unique kind of relation to the substantives of a different category—date, period, point or line—to which they are attached. Such a relation, like that of characterisation, is unique;—in the sense that one of its terms necessarily belongs to a certain category and the other to a certain other category. The relations ‘occupying’ and ‘occurring at’ further resemble the characterising tie in being unmodifiable; thus, of any given date and any given event the only relevant assertion that can be made is that the event either did or did not occur at that date. In saying of an event that it occurs at a certain date or of a material body that it occupies a certain region, the predications may be, not ultimately analysable into definable relations to a definable period or region, but regarded rather as adjectivally unanalysable. ‘Occurring at’ and ‘occupying’ are therefore properly speaking *ties*. It is not, however, formally incorrect to regard them as relations, in the same way as we have allowed characterisation to be analysed as a relation involving the two correlatives characterising and characterised by. From this discussion it will be seen that I incline to the view that instants of time and points of space, as well as time

and space as wholes, are not substantival or existential but merely adjectival.

Now, under the head of the relativity of time and space two distinct philosophical problems are often confused. The view that position in space or time is definable not as absolute, but only as relative to other points or instants is to be distinguished from another view according to which temporal and spatial relations are relations, not between entities such as points and instants, but between what occupies the points or instants. The first of these two problems is appropriately described as the question of the absoluteness or relativity of time and space; the second as the question of the substantival or adjectival nature of time and space. In the *Principles of Mathematics* Mr Russell explicitly maintains the absolute view as regards both these problems; he deliberately asserts that position—a term conveniently used both for space and time—is absolute and not merely relationally definable in terms of other points or instants; and also that points and instants are existents. Now, in the foregoing analysis, I have taken the relative i.e. adjectival view on the second of these two problems, while not rejecting the absolute view on the first. The adjectival view of space and time, in which we deny such separable entities as instants and points, must not be confounded with the class-view: that identity of dating merely means being comprised in a certain assigned class of contemporaneous events. For, in holding that ‘occupying a certain instant’ is an unanalysable adjectival predicate, we maintain at the same time that, *quâ* predicate, it is an identifiable entity, in the same way as the adjective

'red' is an identifiable entity when predicated now of this patch and then again of some existentially other patch. This view is not inconsistent with my previous analysis; for I have repeatedly maintained, particularly in my analysis of the principle of abstraction, that adjectival identity cannot be resolved merely into membership of a certain definable class. My contention for the adjectival nature of space and time amounts to the statement that instants and points are substantival myths. It is not necessary, however, for the purposes of this exposition, to press the question of the substantiality of time and space, for any difference of view on this point does not affect the further development of the subject.

§ 8. Having shown the analogies between the three kinds of stretches—qualitative, temporal, and spatial or rather linear—we will now compare such extensive wholes with classes considered in extension, which may be called *extensional* wholes. It is not a mere accident of language that the term *extension* has two applications in philosophy, these generally occurring in such different contexts that they are not confused. But it is worth while drawing attention to the double use of the word; and, in so doing, to examine a topic, prominent in modern mathematics, concerning the formal agreements and differences between extensional and extensive wholes. An extensional whole, otherwise a class, is naturally associated with the notion of assignable items of which the class is composed; on the other hand, a linear whole—which illustrates an extensive whole of the simplest kind—is apprehended as a whole without thinking of the points it contains. In other words:

the items in an extensional whole are prior in thought to the whole, which appears to be the product of a constructive process; while the extensive whole is prior to any conception of points, which seem to be the result of a similar, but reversed, process of thought-construction. This psychological distinction Mr Russell seems to regard as philosophically negligible; and he devotes a large part of his exposition to a proof of the essential sameness in nature of extensional and extensive wholes. This question raises the same problem as that discussed by Hume and Kant—the former in his quarrel with the mathematicians, and the latter in his solution of the antinomies.

Let us then examine what common-sense would elicit from a consideration of these two kinds of wholes. With regard to extensional wholes, I have adopted the term 'comprise' to represent the relation of a class to any of its items or members, and 'include' to represent the relation of a genus to any of its species; and it is of the first importance to note that, for extensive wholes, an analogous distinction holds between the relation of a line to any of its *points* and the relation of a line to any of its *parts* which are themselves linear. For just as a class comprises items which have to one another the sole relationship of otherness, so a line comprises points which have to one another the sole relationship of otherness; and again, just as members of a species are members of the genus, so points in a linear part are points in the linear whole. Further, since a line or stretch contains parts in the same sense as a genus includes species, it follows that such purely logical or formal relations as overlapping, includent, excludent,

which apply to classes and not to items, apply also to the parts of a stretch but not to points. Now the parts into which a three-dimensional space can be divided are three-dimensional, and have, *quâ* three-dimensional, all the properties of the whole; similarly the parts of a two-dimensional space are two-dimensional; and the parts of a one-dimensional space one-dimensional. On the other hand, of the contiguous parts of a three-dimensional whole the common boundary is two-dimensional; of the contiguous parts of a two-dimensional whole the common boundary is one-dimensional; and of the contiguous parts of a one-dimensional whole the common boundary is zero-dimensional, i.e. a point. Restricting our discussion to the last case, we note a very substantial difference between extensive wholes and extensional wholes; for within a merely extensional whole there are no relations of contiguity, whereas every extensive whole is apprehended as containing parts which are either literally or metaphorically further from or nearer to one another. Hence the notion of a point as a boundary comprised in neither or in both of the two parts of a line has no analogy amongst members of a genus which belong either to one species or to another and cannot belong to both. It further follows that an extensive whole resembles a serial or ordered set of items rather than a mere unordered class or enumeration.

§ 9. Having considered the nature of an extensive whole, i.e. that of which extensive magnitude may be predicated, we will pass to the consideration of the kinds of entities of which distensive or intensive magnitude can be predicated. By distensive magnitude is

meant degree of difference, more particularly between distinguishable qualities ranged under the same determinable<sup>1</sup>. Thus the difference between *red* and *yellow* may be greater or less than that between *green* and *blue*; and similarly the difference between the pitches *C* and *F* may be greater or less than that between *B* and *G*. The notion of difference is apt to be associated with the arithmetical process of addition, for which the term 'addendum' or 'subtrahend' may be used in order to distinguish it from a distensive magnitude. Thus it is preferable to say that successive terms forming an arithmetical progression are obtained by a constant addendum, just as those forming a geometrical progression are obtained by a constant multiplier. This reference to arithmetical and geometrical progressions is needed because the measure of qualitative difference, in its logical and even its philosophical sense, is in some cases or on some grounds to be conceived as an addendum, and in other cases or on other grounds as a multiplier. For example, if a series of colours are presented as in the spectrum, in a continuous spatial order, we might conceive the magnitude of difference between any one hue and any other to be proportional to the length in the spectrum between the two hues. In this case, by taking any hue as origin, say *O*, such that *A* is between *O* and *B*, and representing the difference between *A* and *B* by the symbol *AB*, we should assume that its value was given by the equation  $AB = OB - OA$ . On the other hand, as regards the scale of pitch, the scientist would naturally connect the pitches with the physical process of aerial vibration, and measure each

<sup>1</sup> See Part I, p. 191.



pitch by the number of vibrations per second. On this assumption the 'difference' between  $C$  and  $G$  would be represented by the ratio of  $\frac{3}{2}$ , and that between  $G$  and  $B$  by  $\frac{5}{4}$ , and therefore the 'difference' between  $C$  and  $B$  would be  $\frac{3}{2} \times \frac{5}{4} = \frac{15}{8}$ . These two examples of the two natural modes of estimating degrees of qualitative difference—viz. by an addendum or by a multiplier—are typical of all problems regarding distensive or even intensive magnitudes.

It will be important, however, to contrast either of these more physical modes of conceiving distensive magnitude with the mode that has become familiar to psychologists ever since Fechner's and Weber's experiments. According to Fechner it would appear that the magnitude of difference between the qualities or intensities of sensations should be determined by taking as unit-difference that which is just discernible in an act of perception directed to the sensations as experienced. It should be here noted that we are measuring psychical entities, and not, as in the previous discussion, their physical correlates. Fechner adopted the view that the proper sensational magnitude, either of qualitative or of intensive difference, was obtained by addition, in which equal units were those which were just perceptible. When he compared the resulting sensational magnitude with the magnitude of the stimulus as measured physically, he concluded that, while the sensations could be ranged in arithmetical progression, the corresponding stimuli would form a geometrical progression. By means of an elementary mathematical process it will be seen that this formula can be expressed by saying that the magnitude of the sensation

varies as the logarithm of the stimulus. But this technical development is not our concern here. I wish rather to draw attention to the extraordinary, and in my view baseless, assumption that the just discernible differences at the different points in a scale should be taken to indicate equal *addenda*. If he had assumed what appears to be more plausible that the just discriminable qualities were those which bore a common ratio to one another, the experimental results of Fechner or Weber would have been most naturally expressed in the formula that the sensational magnitude ( $\sigma$ ) varies in proportion to the magnitude of the physical stimulus ( $s$ ) measured from a certain constant ( $s_0$ ): i.e.

$$\sigma = k(s - s_0),$$

where  $k$  is constant; instead of by the formula

$$\sigma = k \log (s - s_0).$$

So far from taking discriminability as equivalent to an addendum, it is more plausible to consider it as equivalent to a ratio<sup>1</sup>. For example, taking the visual magnitudes of four objects  $A$ ,  $B$ ,  $C$ , and  $D$ , if we can just discriminate between the magnitudes of  $A$  and  $B$  and also between those of  $C$  and  $D$ , then it is reasonable to infer that the ratio of  $B$  to  $A$  is equal to the ratio of  $D$  to  $C$ , rather than that the addendum by which  $B$  exceeds  $A$  is equal to the addendum by which  $D$

<sup>1</sup> I am not here concerned with the accuracy of the experiments made by Fechner, nor with his right to make the very wide induction from the artificial nature and limited number of cases that he and his successors have examined. I am referring merely to a logical and not to a psychological question, namely, the justification for regarding our power of discrimination as equivalent to our power of perceiving additions of magnitude rather than ratios.



exceeds *C*. This principle for measuring psychical magnitude may be applied not only to cases of direct sense-perception, but also to those in which we are guided by general psychological considerations; for example, the difference of pleasure that we conceive to be produced by two different increases of income such as that from £100 to £200 and from £1000 to £1100 would not naturally be taken to be equal; the increase from £100 to £110 would rather be considered the equivalent of the increase from £1000 to £1100.

§ 10. We now pass explicitly to the third fundamental kind of magnitude, namely intensive, which has received considerable philosophical attention. Kant regarded intensity as so to speak equivalent to existence or reality, so that that which has greater intensity has for him greater reality. The point in which this view agrees with the modern theory is that intensity has a terminus in the value called zero; and it is in this respect that the distinction between distensive and intensive magnitudes is most clearly marked; the minimum or zero of distensive magnitude is *identity*, whereas the minimum or zero of intensive magnitude is *non-existence*. Another obvious distinction between the two kinds of magnitude is that distensive magnitude is a relation between determinates under some one given determinable, whereas intensive magnitude holds within each separate determinate, or even amongst different qualities under the same determinable. Thus, with regard to the comparative brightness of different hues, we may predicate equal to, greater than or less than, and so also with regard to the loudness of sounds of different pitch. It is impossible, however, to compare

two kinds of intensive magnitude such as the brightness of a light sensation with the loudness of a sound sensation; all we can say is that a colour of zero brightness would be non-existent, and a sound of zero loudness would be non-existent. The subtle point then arises whether the notion of zero-intensity of sound is distinguishable from the notion of zero-intensity of light. In popular language we might ask: Is there anything to distinguish absolute silence from absolute darkness? I think that apart from an organ of sensation having potentialities as a medium for receiving sensations we must say that zero-intensities are indistinguishable; it is only through the capacity of visual and auditory imagery, and indirectly through the possession of organs for conveying these two corresponding kinds of sensation, that distinctions between zeros can have for us any import.

§ 11. In conclusion I have to explain why distensive magnitudes have been confused on the one hand with extensive and on the other hand with intensive magnitudes. As regards the former, the confusion is due to identifying the distensive magnitude of difference, say between the pitches *C* and *G*, with the stretch including all the intermediary pitches. This stretch illustrates what we have called an extensive whole; and, in so far as it can be measured, its measure would be equivalent to that of the difference between *C* and *G*; i.e. its *measure* would be equivalent to that of a distensive magnitude, but the *natures* of the two are non-equivalent. As regards distensive and intensive magnitudes, these agree in so far as they both apply to qualities, and not obviously to things occupying a

quantum of space or time or forming a linear or temporal series; but it is necessary to distinguish them inasmuch as distensive magnitude requires the fundamental conception of different qualities which are yet comparable; while intensive magnitude requires—what has sometimes been paradoxically described as the conception of a thing as merely qualitative, and yet as susceptible of quantitative variation.

§ 12. Having distinguished different kinds of magnitude, we have now to consider how magnitudes of any given kind are to be compared; and we will begin by the simplest kind of magnitude, viz. that which can be predicated of a linear whole.

Mr Russell deliberately adopts the view that the ultimate parts of a line are points, of which the number may be assumed to be  $2^{\text{exp } N}$ , whatever be the magnitude of the line. In other words, any comparison of one line with another in regard to magnitude depends upon something other than the number of points which the lines contain. Hence the magnitude of an extensive whole, as illustrated by a line, cannot be estimated in terms of pure or abstract number. In this respect it is of a totally different nature from a class, the magnitude of which is entirely determined by the number of items it comprises, or by the number of exclusive sub-classes into which it may be divided. It follows then that magnitude, when applied to an extensive whole, has a different meaning from magnitude when applied to an extensional whole. For what I have called an extensive whole Mr Russell uses the term 'divisible whole,' because the notion of dividing is essential to our conception of the relation of part to

whole, particularly in temporal and spatial applications. But, in discussing the principle required for comparing the magnitude of one line with that of another, he uses the phrase '*magnitude of divisibility*.' This phrase appears to me unfortunate inasmuch as it conveys no meaning: entities may be distinguished according as they do or do not possess the quality of divisibility; and the term magnitude is of course required when we discuss whether one thing is greater or less than another. But I fail to see how we can regard one line as being greater than another on the ground that it possesses the quality of divisibility in a higher degree. It is quite certain that the number of parts into which a shorter line can be divided is exactly the same as the number of parts into which a longer line can be divided; as also are the number of points in the one and in the other. The term '*magnitude of divisibility*' therefore appears to me merely to conceal what really is the problem involved in comparing things having extensive magnitude; namely the conception of equality of magnitude.

§ 13. What do we mean by the question, or how can we test, whether one given line or surface or bounded three-dimensional figure is greater or less than another? Or again whether one stretch of hue or of pitch is equal to or greater than another? In general, for two extensive wholes  $M$  and  $N$  of the same kind, if  $M$  includes but is not included in  $N$  it will be agreed that the magnitude of  $M$  is greater than that of  $N$ ; or briefly, the relation of superincident to subincident, whole to part proper, entails the relation of greater to less. But, if the wholes  $M$  and  $N$  are *coexclusive*, then no such test of equality or inequality can be directly

applied; and in order to compare their magnitudes in this case, we must be able to find parts of  $M$  that can be equated to one another as also to parts of  $N$ . This would provide us with a unit magnitude, in reference to which the magnitudes of  $M$  and  $N$  could be numerically compared. If we further assume that the wholes satisfy the strict criterion of continuity as defined by Cantor, then the series of numbers rational and irrational will provide means for comparative measurement of all such magnitudes. On this assumption the only problem that remains is the provision of a test or definition of equality amongst unit parts. The possibility of such a test must be separately examined for the three cases of spatial, temporal and qualitative stretches. As regards spatial wholes of one, two or three dimensions, the classical method is that of superposition, the validity of which must be carefully considered. It is obviously absurd to think of the parts of space themselves as moving; and hence the so-called method of superposition can only have practical significance when we distinguish the material occupant of a place from the place which it occupies. When the material occupants of space are superposed one upon another, and the boundary of one is coincident with that of the other, they are said to be *conterminous*; and when the boundary of one is subincident to that of the other, they may be said to be partially conterminous. Thus the outer boundary of a liquid and the inner boundary of a closed receptacle which it fills are coincident; and, in this case, the volume occupied by the liquid is equal to the volume unoccupied by the receptacle. Again, if any two bodies have a common two-dimen-

sional boundary (which does not enclose a volume), then the boundary of the one has the same *areal* magnitude as that of the other; and, if two bodies have a common one-dimensional boundary, then the boundary of the one has the same *linear* magnitude as that of the other. If, moreover, several different bodies can in either of these three ways be made conterminous with some one given body, the volume, area or length of the corresponding boundary of the one is equal to that of the other. But this predication of equality assumes that the volume of the receptacle, or of the areal or linear boundaries of the superposed bodies remains unchanged; and the assumption that in the course of time a material body does not change its spatial magnitude is in general invalid; hence there is no literally logical justification for asserting equality or inequality in general, either with respect to the same body in different places, or with respect to the different places which the same body may occupy. Science in this case relies upon the constancy (under unchanged conditions) of the volume of *certain* bodies, and uses these as standards by which the changes of volume of *other* bodies are tested. In this process, we are continually acquiring more precise knowledge of causal conditions; but the final justification for comparisons of spatial magnitude is to be found in the coherency or consistency with which the systematisation of measurements and the construction of physical laws can be developed. The conclusion follows then that no directly logical test can be found, and we must be satisfied with the indirect principle according to which comprehensive universals are asserted on the mere ground that they do not lead to appreciable inconsistencies.

The problem of temporal magnitude, like that of spatial magnitude, is met first by the axiom that events that are conterminous at both ends have the same temporal magnitude, and secondly by the postulate that *under identical causal conditions* equal changes occupy equal lengths of time. We then employ some physical process, such as the movement of the hands of a watch, in which the mechanical conditions can be estimated with the closest approximation to exactitude, and adopt as standard time-units the times occupied by the changes thus effected. Conversely, where equal changes are effected during unequal times, we infer that the causal conditions are not identical. In all temporal changes, the means by which we can measure such changes as *equal*, itself depends upon the assumption that we can measure certain spatial, distensive or intensive magnitudes.

Turning now to qualitative magnitudes, we have to consider by what method stretches of hue and pitch can be quantitatively compared. If we agree that the stretch from *A* to *E* is equal to that from *C* to *G* in a scale of pitches, this cannot be tested by any such method as that of superposition, for there is no distinction here corresponding to that between the place which is occupied on the one hand, and that which is movable and can occupy indifferently one place or another on the other hand. If a qualitative stretch has magnitude, this involves the assumption that stretches of the same kind are comparable as greater or less. But how much greater, or by what ratio the two are to be compared must be determined, if at all, by some principle totally different from superposition. Mathematicians who have

written on this subject appear to agree in the view that two magnitudes may be comparable as greater or less, and yet not measurable in terms of number. But, if two stretches are mutually excludent, I can see no sense in which they can be compared as greater or less, unless we have a test of equality; and, when such test is forthcoming, a numerical measurement seems to me immediately to follow. Numerical measurement is not a merely arbitrary one-one correlation between numbers and magnitudes: for such correlation could only mean that for the greater magnitudes we apply higher numbers, and the precise numbers which we correlate would be absolutely arbitrary. Hence it appears to me that if a specific one-one numerical correlation has an objective ground, according to which it is to be preferred to any other, this must be because we have adopted some principle for determining a correct quantitative unit. For example, if we prefer the absolute measurement of temperature to the thermometric measurement as determined say by the changes of volume of mercury, this is because we believe that the differences of temperature indicated by the former scale *do* correspond to really equal differences of magnitude, whereas the other does not. Readers of Clerk Maxwell's *Heat* will learn that the *absolute* measurement of temperature depends upon measurements of heat and work, which are complex quantities, being partly extensive and partly intensive. In all such cases, where we cannot *directly* measure a cause or an effect, we measure it *indirectly* in terms of its effect or cause (as the case may be).

§ 14. The entities of which either extensive, distensive or intensive magnitude can be predicated alone may be termed simple or simplex, and from these kinds of entity we now pass to those which may be called compound or complex, on the ground that two or more magnitudes are combined in our conception of the quantity of the resultant complex. These latter may be illustrated by light sensations, which vary intensively according to their brightness, distensively as regards their hue, and in yet a third respect according to the proportion in which the chromatic and achromatic factors are combined to produce different degrees of saturation. Similarly sound sensations vary intensively according to their loudness, distensively as regards their pitch, and as regards timbre or klang-tint in accordance with the proportional intensities of their constituent tones, under-tones and over-tones. It is convenient to speak, then, of light and sound sensations as three-dimensional, in the sense that there are three distinct determinables under which any such sensation can be defined and quantitatively estimated. But the simplest case of a three-dimensional quantity is space. In space we may take three arbitrary directions; and, according to the ordinary view, the magnitudes (i.e. lengths) along these directions have the unique characteristic of being comparable. Any point in a space of three dimensions is therefore assignable by three ordinates drawn in determined directions from a given point as origin. In this way a surface in three dimensions, or a line in two dimensions, differs from what is called a *graph*, in that the magnitudes represented by the ordinates of a point in the graph are of different kinds and therefore incom-

parable. For example: a graph representing the covariation of work done and hours expended uses two incomparable magnitudes.

The general topic which we have now to consider is that of a derived quantity that is constructed by some kind of combination of other quantities. In constructing a quantity comparable with each of those combined, the processes of addition and subtraction can alone be applied; and, conversely, addition and subtraction can only be applied to comparable quantities. Such addition and subtraction may be termed *concrete*, in antithesis to *abstract* in which pure numbers are concerned whose sum or difference is also a pure number. Now I shall maintain that processes analogous to multiplication and division may be employed in constructing a quantity of a different kind from any of those that are combined in its construction; and such multiplication or division may also be called concrete. Thus, considering first the three notions of length, area, and volume, I shall say that the multiplication of two differently directed lengths constitutes an area, and that of three differently directed lengths constitutes a volume. Here we are extending the operation called multiplication beyond its primary use. For, while it is universally agreed that we may multiply a pure number by a pure number, in constructing another pure number, or a quantity of any kind by a pure number, in constructing a quantity of the same kind, yet most mathematicians have refused to allow that by multiplying one quantity by another we may construct a third quantity different in kind from both the quantities multiplied. They maintain that what is multiplied is *the numerical measure* of the quantities



and not the quantities themselves. Similarly with regard to division: it is agreed that we may divide a pure number by a pure number in constructing another pure number, or a quantity of any kind by a pure number in constructing a quantity of the same kind; but we are prohibited from dividing one quantity by another in constructing a quantity different in kind from the quantities divided. In this case too, the so-called division is regarded as a division not of the quantities but of their numerical measures. My first objection to this view is that it offers no means of distinguishing between the multiplication or division of a quantity by a pure number, which yields a quantity of the same kind, from that very different kind of multiplication or division which yields a quantity different in kind from those multiplied or divided. My disagreement, however, with the almost unanimous opinion of mathematicians may perhaps be considered merely verbal; but the view that I maintain is, I think, based upon an important logical principle. Apart from any conception of numerical measurement which adopts numbers, integral, rational and irrational, it appears to me that we must conceive the process of multiplying say a foot by an inch (which involves no idea of number) as a construction by which, from two magnitudes of the same kind, a third magnitude of a different kind is derived. If no such magnitude were presented to perception or thought, it would follow that no meaning could be attached to such multiplication; but, inasmuch as an area is a genuine object of thought construction, I see no insurmountable objection to speaking of the process of multiplication as that by which area, for instance, is constructed out of two

directed lengths, or volume out of three. The mathematicians who reject this idea hold that the notion of *units* of different kinds is sufficient, without introducing the multiplication or division of units. It is agreed that the area of a rectangle whose sides are of unit length is a unit area, and the volume of a cube whose sides are of unit length is a unit volume. In this way the numerical measures of area and volume are obtained by multiplying the numerical measures of their sides; but in my view we must allow that the lengths themselves are multiplied, for otherwise we could not distinguish the different kinds of magnitudes constructed, since where only abstract numbers are concerned only abstract numbers are constructed, and there is nothing to indicate the difference between one quantity thus derived and another. Passing from concrete multiplication to concrete division we have what may be thought a more interesting and certainly a wider application of the same general principle. Those quantities which are derived by dividing one kind of quantity by another may be called rate-quantities, or in certain cases degree-quantities. A rate-quantity is expressed in familiar English by the Latin word *per*, of which it is easy to multiply examples: e.g. space traversed per second, wages earned per hour, pleasure experienced per minute, pressure per square foot, mass per cubic foot. The two constituent quantities in this kind of division may be themselves complex or of different kinds, extensive, distensive or intensive; but so far as the conception of concrete division is concerned, no logical distinctions are required in analysing the *general* notion of a rate-quantity. Each of the rate-quantities constructed by

this species of division is a quantity of a different kind from the quantities of which it is constituted; and, as in multiplication, it is always useful for arithmetical purposes to adopt as the *derived* unit-quantity that which is constructed out of fundamental unit-quantities. The general term 'rate' which I have introduced is in common use: thus we mean by the rate of wages the quantum of wages earned per unit of time, by the rate of speed the quantum of length traversed per unit of time, and by the rate at which pleasure is being experienced, the quantum of pleasure per unit of time—these being cases in which the rate is estimated in reference to time. Again the rate called hydrostatic pressure is the quantum of pressure per unit of area; the rate called density is the mass per unit of volume. The term degree, which is sometimes used instead of rate, is ambiguous inasmuch as it is often used as equivalent to intensity; but the terms rate and intensity or degree ought to be clearly distinguished, because the notion of intensity refers to a single determinate quality, whereas rate is always constituted out of two distinguishable quantities; moreover the notion of rate, which involves concrete division, is always correlated with concrete multiplication. For example velocity, i.e. rate of movement, which involves the division of space by time, involves the converse process of multiplying velocity by time in constructing space. But if we conceive of velocity only by its numerical measure, confusion results between an abstract number on the one hand and the very many different kinds of quantity on the other hand that may be measured by the same abstract number.

§ 15. The practical importance of recognising concrete multiplication and division is best indicated by explaining what is meant by the *algebraical dimensions* of a quantity. We have already spoken of dimensions in its geometrical sense; thus an area is of dimension *two* in regard to length, a volume of dimension *three* in regard to length. Symbolising the dimension length by  $[L]$  that of area is symbolised by  $[L^2]$  and that of volume by  $[L^3]$ . Similarly velocity, i.e. length per time, is dimensionally  $\frac{[L]}{[T]}$  or  $[L] \cdot [T^{-1}]$ ; acceleration, i.e. velocity per time is  $[L] \cdot [T^{-2}]$ ; density, i.e. mass per volume is  $\frac{[M]}{[V]}$ , i.e.  $[M] \cdot [L^{-3}]$ ; momentum, i.e. mass  $\times$  velocity is  $[M][L][T^{-1}]$ ; force, i.e. mass  $\times$  acceleration is  $[M] \cdot [L] \cdot [T^{-2}]$ ; hydrostatic pressure, i.e. force per area is  $[M] \cdot [L^{-1}][T^{-2}]$ , etc., etc. Now the one rule as regards dimensions is that the additions and subtractions that are involved in a quantitative equation must always operate upon homogeneous quantities; i.e. upon quantities all of which have the same dimensions—these dimensions being generally expressed in terms of the three fundamental incomparables *mass*, *length*, and *time*. Regarding multiplication and division, in accordance with my view, as real operations performed upon concrete quantities, the square bracket in the above symbols stands for a *concrete unit*. For example the velocity '320 feet per 60 seconds' means  $\frac{320 \text{ ft.}}{60 \text{ sec.}} = \frac{16 \text{ ft.}}{3 \text{ sec.}} = \frac{16}{3}$  of *unit velocity*. Those mathematicians who hold that such an expression as  $\text{ft.} \div \text{sec.}$  is meaningless have to maintain that the mathematical equations which are

used to express physical facts are concerned only with the numerical measurement of concrete quantities, whereas I hold that they are concerned with the concrete quantities themselves.

§ 16. There is one very unique case in concrete division, viz. where the dividend and divisor are quantities of the same kind. In general the result of such division is to construct a pure ratio, i.e. a magnitude which, when entering as multiplier or divisor of a quantity of any kind yields a quantity of the same kind, like the processes of addition and subtraction of quantities. But when a length is divided by a length, or an area by an area, we often intend the result of such division to represent an *angle*. It is therefore necessary to distinguish those cases in which the division of a length by a length represents a mere ratio, from those in which it represents an angle. In the former case, the quotient being a pure number can be used as a multiplier or divisor for a quantity of any kind whatever; but in the latter case this is never possible; one angle can only be mathematically combined with another angle, and this only by the operation of addition or of subtraction. The further complication in respect of the measurement of an angle is that this measurement may be used in different algebraical applications alternatively either as an abstract ratio or as a concrete quantity, which is denoted by the term angle. But the special question which, in my view, requires a clear answer is how to distinguish the process of dividing length by length that yields a mere ratio, from what appears to be the same process and yet yields an angle. The answer seems to be that when we are merely *comparing* two lengths

which may be said to be *dissociated*, their comparison yields a mere ratio, while when connecting two *associated* lengths in the process of division, we are constructing an angle. Thus, when we define the magnitude of an angle by the ratio of the arc of a circle to its radius, the arc and the radius are associated in our conception of the mode in which the angle is constructed; but when we are merely comparing the length of one line with that of any other, no natural association between the two lines is involved. The same holds of the differential coefficient  $dy$  by  $dx$ , when used in geometry to represent the slope of a tangent of a curve, which is a concrete quantity in the same sense as the quotient foot by second representing velocity.

§ 17. To sum up: Of the different kinds of magnitude, the first division is between abstract and concrete, abstract magnitudes being represented by pure numbers, these falling into the three divisions of integral, rational and irrational. Amongst concrete quantities—namely those that involve conceptions obtained from special kinds of experience, and which are therefore not purely logical—we distinguish the fundamental or primitive from the complex or derivative; the former being subdivided into extensive, distensive and intensive magnitudes, out of which the various derived or complex quantities have been shown to be constructed by operations analogous to arithmetical multiplication and division. These complex magnitudes fall again into different kinds, the distinctions between which may be always indicated by expressing the quantity dimensionally, i.e. as involving a concrete product of different fundamental quantities, each entering with a positive or negative

index. Finally a fundamental distinction has been drawn between addition or subtraction on the one hand and multiplication or division on the other; inasmuch as the quantities added or subtracted must be of the same kind, i.e. represented as dimensionally equivalent; whereas the operations of multiplication and division yield a quantity different in nature from its factors, which, however, together determine its nature. Throughout the whole discussion of concrete magnitudes, the difficult problem of defining or testing equality has been examined for each fundamentally distinct kind of quantity. The treatment has been comparatively elementary, the reader being referred for more subtle distinctions and analyses to works which deal primarily with mathematics and its philosophy.

## CHAPTER VIII

## INTUITIVE INDUCTION

§ 1. INDUCTION in general may be contrasted with deduction in that for a universal conclusion deduction needs universal premisses, whereas in induction a universal conclusion is drawn from instances of which it is a generalisation. Here the emphasis is upon the word instances, because although the customary account of deduction is that the range of the conclusion is identical with that of the narrowest of the premisses, yet deduction must include cases in which the range of the conclusion is not identical with that of any one of the premisses, and may even be wider than the widest of them. Actually the antithesis between inductive and deductive inference is not so fundamental as that between demonstrative and problematic inference; for every form of induction, except the problematic, is based upon the same fundamental principles (and these alone), as syllogism and other forms of deduction; whereas it is impossible to establish a theory of problematic induction, without recourse to certain postulates that are not involved in either form of demonstration, whether deductive or inductive. Now the fundamental principles which underlie demonstrative forms both of induction and deduction are themselves based upon a kind of inference which may be called intuitive induction. This process is not limited to the establishment of the principles of demonstration, but applies also to certain material as well as formal generalisations.

We have so far referred to two types of induction, viz., intuitive and demonstrative; it will be convenient to distinguish in all four varieties, namely intuitive, summary, demonstrative and problematic. Of these the three former will be discussed in the present Part of this work, but problematic induction will be examined in detail in a separate Part, on the ground, specified above, of its dependence upon special postulates.

§ 2. Before treating the main topic of this chapter, we must discuss the necessarily preliminary process known as abstraction, the nature of which was a special subject of philosophical and psychological controversy amongst James Mill and his contemporaries. The discussions of that date started from the supposition that what was presented in our earliest acts of perception was a combination of impressions from different senses, such as those of sight and touch. From this pre-supposition, upon which both parties were agreed, the difficulty was raised as to how the percipient could single out an occurrent impression of one sense from the concurrent impressions of other senses. This pre-supposition, however, is fundamentally mistaken. For, in *fact*, our earliest acts of attention, which yield any product that could be called a percept, are directed to impressions of one sense at one time, and to impressions of another sense at another time. For example, the child when interested in the colour of a ball, is attending to his visual impressions apart from any motor or tactual sensations that he may be experiencing in handling the ball; that is to say, his attention is from the first exclusive, and it is only in further progress of attentive power that his attention becomes inclusive. The atten-

tion that includes visual with tactual impressions is a higher and later process than the attention which is directed either exclusively to the visual impressions or exclusively to the tactual impressions. The fact that we can and do attend to impressions of one order in disregard of concurrent impressions of other orders, explains how our primitive perceptual judgments, from the first, assume a logically universal form. For, in predicating a determinate colour, for instance, of any given impression, there is a recognition that the same determinate can be predicated of *all* impressions which agree with the given impression in respect of colour, however much they may disagree in other respects. Now, if this be granted, it has an important bearing upon another serious historical controversy—namely that between Mill and his opponents as to the foundations of geometry. Both parties to this dispute started with an obscure view, that there was an opposition between intuition and experience; whereas in truth intuition is a *form* of knowledge, in relation to which experience is the *matter*. The intuitionists seem to have held that the intuitive form of knowledge involved no reference to experience; whereas the empiricists forgot, when relying upon experience as the sole factor in knowledge, that knowing is a mode of activity, and therefore not of the same nature as sense-experience which is merely passive or recipient. The truth is that when we have asserted a predicate of a particular, we have apprehended the universal in the particular, in the sense that the adjective is universal and the object of which it is predicated is particular.

§ 3. There is another sense in which we may be



said directly to apprehend the universal in the particular, namely in regard to certain classes of propositions, where the terms universal and particular apply to the propositions themselves, and not to the distinction between the subject and the predicate within the proposition. It is at this stage that we pass, in our discussion, from abstraction to our main topic, viz., abstractive or intuitive induction. The term intuitive is taken to imply felt certainty on the part of the thinker; and it is characteristic of propositions established by means of intuitive induction that an accumulation of instances does not affect the rational certainty of such intuitive generalisations. The procedure by which these generalisations are established may be shown by psychological analysis to involve an intermediate step by which we pass from one instance to others of the same form and in this passage realise that what is true of the one instance will be true of all instances of that form.

§ 4. Two types of intuitive induction may be distinguished, experiential and formal, although these types are not precisely exclusive of one another.

The experiential type of intuitive induction may be illustrated from our immediate judgments upon sense-impressions and the relations amongst them. For example, in judging upon a single instance of the impressions red, orange and yellow, that the qualitative difference between red and yellow is greater than that between red and orange (where abstraction from shape and size is already presupposed) this single instantial judgment is implicitly universal; in that what holds of the relation amongst red, orange and yellow for this

single case, is seen to hold for all possible presentations of red, orange and yellow. Again in immediately judging that a single presented object, whose shape is perceived to be equilateral and triangular, is also equiangular (where abstraction from colour and size is presupposed) we are implicitly judging that *all* equilateral triangles are equiangular. Similarly when judging for a single instance that the sounds *A, C, F*, produced, say, from the human voice, are in an ascending scale of pitch, we are implicitly judging that *all* sounds—apart from differences of timbre or loudness such as those produced by the violin or piano—that can be recognised as of the same pitches *A, C, F*, are also in an ascending order of pitch. The universality of these experiential judgments extends over imagery as well as sense impressions: the fact that we can identify a specific image as corresponding to a specific impression is sufficient to enable us directly to transfer our judgments about the relations amongst impressions to those amongst the corresponding images. These elementary illustrations show that intuited universals about colours and pitches are of the same epistemological nature as those about geometrical figures, in that the judgment upon a single presented instance is sufficient for the establishment of a universal extending in range over imagery as well as impression.

§ 5. Passing now to other experiential judgments, which are not merely sensational, we may illustrate intuitive induction from introspective judgments. For instance, when I judge that it is the pleasure of this or that experience which causes me to desire it, I am implicitly universalising and maintaining that the

pleasure of any experience would cause me to desire it. And again, when I judge that the greater resultant desire for one possible alternative than for any other causes me to will that alternative, I am judging that this will hold for all my volitional experiences. An important sub-class of experiential judgments which are intuitively inductive consists of moral judgments. Thus, when anyone judges that a certain act characterised with a sufficient degree of precision is cowardly, or dishonest, or generous, he is implicitly judging that all acts of the same specific character would be characterisable by the corresponding moral attribute. That this is not a case of mere abstraction is clear when we consider that the characteristics used to define the nature of the action are other than ethical, and that the judgment is therefore synthetic. This intuitive aspect of moral judgments assumes importance as reconciling the two forms of ethical intuitionism to which Sidgwick refers as Perceptual and Dogmatic, the first of which stands for the particular, and the second for the universal, intuition. For, in my view, the Dogmatic form of intuition is not genuinely intuitive except so far as it is based on the Perceptual. Instead, therefore, of distinguishing moralists according to what they hold to be the nature of an ethical intuition, it is more important to distinguish them according as they base their doctrine upon genuinely intuitive judgments, e.g. Kant; or upon judgments accepted on authority as expressions of the voice of God, e.g. Butler.

§ 6. The gulf between experiential and formal intuition is bridged by considering certain intermediary forms of intuitive apprehension in which, according as

the range of universality increases, we depart further from the merely experiential and approach nearer to the merely formal type. A typical case is the merely experiential judgment that red and green cannot both be predicated of the same visual area by one person at one time. The judgment is first universalised when the experient sees that the same holds of all cases of the specific determinates red and green. But this judgment almost immediately passes into the wider universal that any two different determinates under the determinable colour are similarly incompatible. And when lastly the experient extends the range of his judgment to all determinables, he has reached a formal intuition, namely that any two different determinates under *any* determinable are incompatible.

To this formal type of intuition belong all intuitively apprehended mathematical, as well as purely logical, formulae. For instance, the algebraical formula known as the Distributive Law is intuitively reached in some such way as this: perceiving that

$$3 \text{ times } 2 \text{ ft.} + 3 \text{ times } 5 \text{ ft.} = 3 \text{ times } (2 \text{ ft.} + 5 \text{ ft.})$$

we immediately realise that

$4 \text{ times } 7 \text{ days} + 4 \text{ times } 9 \text{ days} = 4 \text{ times } (7 \text{ days} + 9 \text{ days})$ ,  
and in this step we are virtually apprehending the Distributive Law symbolically expressed thus:

$$n \text{ times } P + n \text{ times } Q = n \text{ times } (P + Q)$$

where  $n$  stands for any number, and  $P$  and  $Q$  for any two homogeneous quantities.

A logical example of a similar nature is the formula of the simple conversion of particular affirmative propositions. This is reached by perceiving, for instance,

that 'Some Mongols are Europeans' would imply that 'Some Europeans are Mongols,' and at the same time that

'Some beings incapable of speech have the same degree of intelligence as men' would imply that 'Some beings having the same degree of intelligence as men are incapable of speech.'

This leads to the virtual apprehension of the universally expressed implication:

'Some things that are  $p$  are  $q$ ' would imply that 'Some things that are  $q$  are  $p$ '

where  $p$  and  $q$  stand for any adjective.

§ 7. This example of the establishment of logical formulae by means of intuitive induction has an educational importance in correcting a certain prevalent conception of the function of logic. What is called formal or deductive logic is usually taught by first presenting general principles in a more or less dogmatic form, with the result that the learner is apt to use these principles merely as rules to be applied mechanically in testing the validity of logical processes. Instead of leading him to conceive of these rules as externally imposed imperatives, an appeal should be made to him to justify all fundamental principles by the exercise of his own reasoning powers; and this exercise of power will involve the process of intuitive induction.

## CHAPTER IX

### SUMMARY (INCLUDING GEOMETRICAL) INDUCTION

§ 1. THE term summary induction is here chosen in preference to what, in the phraseology of the old logicians, was called 'perfect induction,' to denote a process which Mill regarded as not properly to be called induction; on the ground that the conclusion does not apply to any instances beyond those constituting the premiss. Mill's contention can certainly be justified inasmuch as the process involves precisely the same logical principles, and these alone, that govern ordinary deduction. In fact, the process of summary induction may be expressed in the form of a syllogism in the first figure. For example:

*Major Premiss.* 'Sense and Sensibility' and 'Pride and Prejudice' and 'Northanger Abbey' and 'Mansfield Park' and 'Emma' and 'Persuasion' deal with the English upper middle classes.

*Minor Premiss.* Every novel of Jane Austen is identical either with 'Sense and Sensibility' or with 'Pride and Prejudice' or with 'Northanger Abbey' or with 'Mansfield Park' or with 'Emma' or with 'Persuasion.'

*Conclusion.*  $\therefore$  Every novel of Jane Austen deals with the English upper middle classes.

Here the enumeration standing as subject in the major premiss is the same as the enumeration standing as predicate in the minor premiss. But, in the former, reference is made to every one of the collection, in the

latter to some one or other. This precisely corresponds to the characteristic of first figure syllogism; namely that the middle term is distributed as subject of the major and undistributed as predicate of the minor. In text-book illustrations of perfect induction the minor premiss is almost invariably omitted, because the illustrations chosen—such as the Apostles or the months of the year—are so familiar that the completeness of the enumeration is assumed to be known by every ordinary reader and therefore does not require to be expressed in a separate minor premiss. The same process is exhibited by an example in which each of the items enumerated is a universal instead of being a singular:

Every parabola and every ellipse and every hyperbola meet a straight line in less than 3 points.

Every conic section is either a parabola or an ellipse or a hyperbola.

∴ Every conic section meets a straight line in less than 3 points.

§ 2. Another case of perfect induction, which has specific bearing upon induction in general, may be expressed symbolically in the following syllogism:

$s_1$  and  $s_2$  ... and  $s_n$  are  $p$ .

Every examined case of  $m$  is identical either with  $s_1$  or with  $s_2$  ... or with  $s_n$ .

∴ Every examined case of  $m$  is  $p$ .

A summary or perfect induction of this form is the necessary preparatory stage in gathering together the relevant instances for establishing an unlimited generalisation. For the *conclusion* thus obtained, constitutes the *premiss* from which we directly infer, with a higher or lower degree of probability, that 'Every case of  $m$  is  $p$ .'

Whewell pointed out the importance and difficulty of discovering 'the concept  $p$  under which the instances are colligated.' He, in agreement with other critics of Mill, accordingly held that the process of induction was completed in the discovery of this colligating concept, on the ground that this process alone required something like genius to perform, while it is the easiest thing in the world to pass from every examined instance to every instance. Mill, on the other hand, considered that this process only supplied the requisite premiss for a genuine inductive inference. To illustrate his view, Whewell had chosen Kepler's famous discovery of the formula for the orbit of the planets, and it was towards this illustration that Mill directed his criticism. Expressed in terms of the above used symbols.

Let  $m$  stand for 'positions of a certain moving planet,'  
 $s_1, s_2 \dots s_n$  ,, ,, 'the several observed positions,'  
 and  $p$  ,, ,, 'being a point on a certain ellipse.'

The syllogism which expresses the process of perfect induction used by Kepler will then be as follows:

'Each of several observed positions is a point on a certain ellipse.'

'Every examined position of a certain moving planet is identical either with one or with another of these several observed positions.'

∴ 'Every examined position of the moving planet is a point on that ellipse.'

This formula had not been discovered by any previous astronomer, and, on the grounds already assigned, Whewell maintained that the discovery constituted the completion of the induction. To this Mill demurred, because by induction he meant a process in which the

conclusion is an unlimited universal extending beyond examined instances; he, however, failed to observe that Kepler had actually gone beyond the examined instances and had described the *complete* orbit of the planet by inferring that what held of the examined positions would hold of all the interpolated positions. Kepler had thus unconsciously made a genuine induction in the sense required by Mill. Whewell was concerned with the art of discovery, and therefore held that the essential factor in induction was the discovery of the colligating concept; whereas Mill was concerned with the science of proof, and therefore held that the essential factor in any induction (that was not merely formal or demonstrative) was the inferential extension from examined to unexamined instances.

§ 3. Having illustrated the process of summary (or perfect) induction by familiar examples, in which the conclusion applies to a finite number of cases which are enumerable, we proceed to consider a more interesting type of summary induction in which the conclusion applies to an infinite number of cases which are non-enumerable. This type occurs in geometrical proofs of geometrical theorems, and has been more or less confused on the one hand with merely intuitive, and on the other hand with problematic induction. It differs, however, from the former in that its conclusion cannot be reached from an examination of one or of a few instances; and from the latter in that the conclusion does not extend beyond the range of the examined instances—these being apprehended in their infinite totality.

It is well known that there are two modes by which geometrical theorems may be proved, viz. 'analytical'

and 'geometrical.' Strict analytical proof has the same logical character as *algebraical* proof, and comes under the head of functional deduction. Such proofs do not require the aid of geometrical figures. But the *geometrical* method of proof depends essentially upon the use of such figures. It may further be pointed out that the analytical method has an indefinitely wider scope than the geometrical. For example, by employing mere analysis we can construct spaces of various different forms other than Euclidian; and certainly a geometrical method would be impossible except as applied to our space which is presumed to be Euclidian. The actual procedure in constructing any non-Euclidian space is to bring forward some four or five axioms which must be (a) independent of one another, and (b) mutually consistent. These axioms, however, are not put forward categorically, but purely hypothetically; it follows, therefore, that the theorems which, for convenience are said to be deduced from the axioms, should be more strictly said to be implied by the axioms. Such systems, therefore, are throughout implicative and not inferential. In other words, a supposed space, definable by any chosen set of axioms, would have such and such other characteristics which these axioms would formally imply. On the other hand, the geometrical method is a method of proof or inference, inasmuch as we accept its conclusions as true only because we have accepted its axioms as true.

§ 4. We must therefore examine the process by which the axioms of geometry are established. These, *quâ* axioms, are not reached by deduction; and, since they are universal in the specific sense that they apply



for an infinite number of possible instances, it would seem that some form of induction is required for their establishment; unless we adopt the view that they are obtained by a process which embraces all possible cases in a single act of direct intuition. This latter appears to be the view of Kant who held, as regards geometry, that our intuitions are from the first universal, and that they therefore function as premisses for deducing any, or any other, given case.

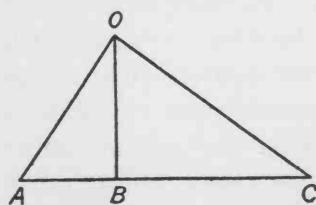
In order to examine this question let us take the familiar axiom conveniently expressed in the form: 'Two straight lines terminating at the same point cannot intersect at any other point.' This is the most important axiom which does not hold of non-Euclidian spaces in general. Independently, however, of the nature of any other kind of space, the axiom certainly represents the manner in which we actually intuit *our* space, whether falsely or truly. Now this axiom, in its universality, can be established only by means of imagery and not by mere perception; for the compass over which the axiom holds is beyond the range of actual perception. For in the first place it is only through imagery that we can represent a line starting from a certain point and extending indefinitely in a certain direction; and, in the second place, we cannot represent in perception the infinite number of different inclinations or angles that a revolving straight line may make with a given fixed straight line. We may, however, by a rapid act of ocular movement represent a line revolving through  $360^\circ$  from any one direction to which it returns. In this imaginative representation the entire range of variation, covering an infinite number of values, can be exhaus-

tively visualised because of the continuity that characterises the movement. It is only if such a process of imagery is possible that we can say that the axiom in its universality presents to us a self-evident truth. It is therefore this species of summary induction that is employed to establish geometrical axioms—differing, as explained above, on the one hand from mere intuitive induction, inasmuch as one or a few specific cases would not constitute an adequate premiss; and, on the other hand, from induction in Mill's specific sense, since the conclusion does not go beyond the premisses taken in their totality.

§ 5. I shall further maintain that if, in the course of a geometrical proof which may involve several successive steps, the perception or image of a figure is required for any single step, this is because we have to go through precisely the same process of summary induction, embracing an infinite number of specialised cases of which the figure under inspection is one—all of these being included in the subject of the universal conclusion to be proved at that step. Speaking generally, in any one demonstrative step, the major premiss is a universal previously established, and from this universal major it is required to establish a new universal conclusion. It is obvious that this can only be done by means of a universal minor; and it is in the establishment of the universality of the minor that consists the logical function of the figure. The arbitrarily chosen figure under inspection can only be used as a minor term to prove the conclusion about that single figure; and hence, to obtain the required universal conclusion, the minor must be universalised by the same logical process that

is used for establishing the explicit axioms. Now the Euclidian geometry might have been established by purely analytical methods; provided first, that a sufficient number of axioms had been explicitly formulated; and secondly that each of these axioms had been established for itself by the process of summary induction. Such an analytical system would dispense with the use of figures as objects either of perception or of imagery *in the course of the proof*, these being only required in the process of establishing the axioms themselves.

To show by specific illustration how the geometrical proof uses a figure, we will select a very frequently assumed, but not explicitly stated axiom, which, in Euclid's proofs is required to supplement the explicit axiom 'the whole is greater than its part,' or more precisely, 'the whole is equal to the sum of its parts.' Before this explicit axiom can be used, we must be satisfied that the two elements of the figure, one of which is to be greater than the other, do stand in the relation of whole to part. The axiom to which I refer is actually employed by Euclid and most geometers in the propositions numbered 5, 6, 7, 16, 18, 20, 21, 24, and 26 in Euclid, Book I. It may be formulated as follows:



follows: The angle subtended at any point by a part of a line is part of the angle subtended by the whole line.' If the reader is not familiar with this new axiom, he must go through a process in which he imagines a line revolving in a plane through a point ( $O$ ) from some initial direc-

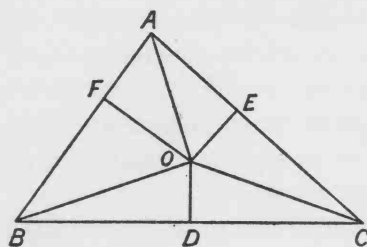
tion ( $OA$ ) to a final direction ( $OC$ ), so that it will intersect the whole line ( $AC$ ) in a series of successive points. In this way, and in this way alone, can he accept the universality of the required conclusion that the angle  $AOC$  is greater than the angle  $AOB$ . In Euclid's theorems enumerated above it will be found that this axiom is required in every case to establish the conclusion that a certain angle is greater than another; and that this conclusion is a necessary step in the further progress of each proof.

Geometrical induction involves, in addition to the summary process above explained, two further processes which are of the nature of intuitive induction, as explained in the preceding chapter. Of these two, the first is concerned with absolute position, the second with absolute magnitude. Thus, having reached a universal by summary induction limited to figures occupying a certain position, it is by intuitive induction that we pass to figures of the same specific shape and magnitude occupying any other possible position; and again from a figure imaged as having a certain magnitude, to figures of the same specific shape but of any other possible magnitude. I have described these two processes as of the nature of intuitive induction, in which we universalise by abstracting from variable position and from variable magnitude; but they might otherwise be regarded as involving the conception of position and magnitude as being—not absolute—but relative to the percipient's own position and to his distance from the figure depicted in imagination.

§ 6. Having illustrated the proper use of the geometrical figure, we shall proceed to illustrate what may

be called its abuse ; and give, by means of a figure, an alleged proof that every triangle is isosceles :

*To prove that every Triangle is Isosceles.*



Let the bisector of the vertical angle  $A$  meet the perpendicular bisector of the base  $BC$ , whose middle point is  $D$ , at the point  $O$ . Join  $BO$ ,  $CO$ , and draw  $OE$  perpendicular to  $AC$ , and  $OF$  perpendicular to  $AB$ . Then,

- (1) the triangles  $BOD$ ,  $COD$ , are congruent; for  $BD = CD$ ;  $OD$  is common; and  $\angle BDO = \angle CDO$ ;  
 $\therefore BO = CO$ .
- (2) the triangles  $AOE$ ,  $AOF$ , are congruent; for  $AO$  is common;  $\angle OAE = \angle OAF$ ; and  $\angle OEA = \angle OFA$ ;  
 $\therefore AE = AF$  and  $OE = OF$ .
- (3) the triangles  $COE$ ,  $BOF$  are congruent; for, by (1)  $CO = BO$ ; and by (2)  $OE = OF$ ; and hence  $CO^2 - OE^2 = BO^2 - OF^2$ ;  
i.e. (since  $CEO$  and  $BFO$  are rt  $\angle$ )  $CE^2 = BF^2$ .  
Hence, by (2)  $AE = AF$  and, by (3),  $CE = BF$ ;  
 $\therefore$  by addition  $AC = AB$ . Q.E.D.

Here we see that the axiom : 'the whole is greater than its part' is used in its more precise form, 'the whole is equal to the sum of its parts.' Now before we can state as regards the straight line  $AFB$ , that

$$AB = AF + FB,$$

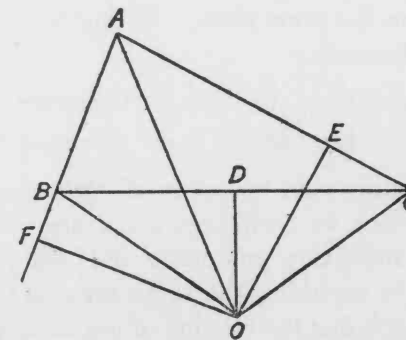
we must be sure that  $AF$ ,  $FB$  are really parts of  $AB$ ;

whereas if  $F$  was beyond  $AB$ , then  $AF$  would be the whole and  $AB$ ,  $BF$  would be its parts.

The fallacy incurred in this proof arises from the mistaken intuition that the bisector of the vertical angle  $A$  meets the perpendicular bisector of the base  $BC$  at a point  $O$  *inside* the triangle. By drawing an incorrect figure and thus convincing ourselves of the false conclusion, we had unconsciously universalised from the figure before us that for every case the two bisectors would meet at a point within the triangle, this being indicated in the figure as drawn. In other words, we have swallowed the relation presented in the drawn figure as being universalisable, without having gone through the necessary summary induction.

We may proceed to draw the corrected figure. From this we reach, as before, the two conclusions

$$AE = AF \text{ and } CE = BF.$$



But now we see that

$$AC = AE + EC,$$

while

$$AB = AF - BF.$$

Euclidian demonstration professes to be based on pure reasoning, in such a manner that the figure may be drawn quite inaccurately, and yet the force of the proof be equally cogent. But it may happen, as in the case before us, that the figure is drawn with a degree of inaccuracy which affects the proof; because the particular demonstration, involving unconscious reference to the figure drawn, has been illegitimately universalised.

§ 7. My explanation of the logical function of the figure in geometrical demonstration differs fundamentally from that put forward by Mill, who maintains that it is by *parity of reasoning* that what is apprehended to be true for the one drawn figure, is apprehended to be true for any other figure (within the scope of the conclusion). But the passage from the demonstration for one case to that for any other case can only be said to exhibit 'parity of reasoning' when the two demonstrations have the same *form*. Taking for example the two demonstrations:

(1) every  $m$  is  $p$ ; this  $S$  is  $m$ ; therefore this  $S$  is  $p$ ;

(2) every  $m$  is  $p$ ; that  $S$  is  $m$ ; therefore that  $S$  is  $p$ ;

we may certainly pass by parity of reasoning from (1) to (2) inasmuch as both arguments are of the same form, the words 'this' and 'that' indicating difference in matter. In ascribing the same form to (1) and (2), what is meant is that the relation of implication between the premisses and conclusion of the one is the same as that between the premisses and conclusion of the other. But in order to use an implication for the purposes of inference, we should have to *assert* 'This  $S$  is  $m$ ' for

case (1) and 'That  $S$  is  $m$ ' for case (2); for although the relation of implication is the same in the two arguments, it does not follow from having asserted the minor of the one that we can, on this ground, assert the minor of the other. Now, in order to establish the required conclusion 'Every  $S$  is  $p$ ' we must first establish the universalised minor 'Every  $S$  is  $m$ .' No reasoning process (in the accepted meaning of the term) would enable us to pass from the case 'This  $S$  is  $m$ ' to 'That  $S$  is  $m$ ' and to 'That other  $S$  is  $m$ ' *ad infinitum*; and the only mode of establishing the required universal minor 'Every  $S$  is  $m$ ' is through some process of induction, the nature of which we have been describing.

## CHAPTER X

### DEMONSTRATIVE INDUCTION

§ 1. HAVING so far examined intuitive and summary induction, we now pass to the third type of inductive inference distinguished at the outset, namely demonstrative induction. As its name suggests, this form of inference partakes both of the nature of demonstration and of induction. It includes several different forms, the characteristics common to them all being (1) that they are demonstrative, in the sense that the conclusion follows necessarily from the premisses; and (2) that they are inductive, in the sense that the conclusion is a generalisation of a certain premiss or set of premisses which, taken as a collective whole, may be spoken of as 'the instantial premiss.' The possibility of arriving demonstratively at a conclusion wider than the premisses, depends here upon the nature of the major premiss, which is not only universal but composite. In short demonstrative induction may be described as that form of inference in which one premiss is composite and the other instantial; the conclusion being a specification of the former and a generalisation of the latter.

§ 2. In explaining the nature of demonstrative induction as above described, the composite nature of the major premiss brings us back to those fundamental modes of inference specified in Part I, Chapter III on compound propositions. There  $P$  and  $Q$  are taken to stand for any propositions, and four composite relations

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are distinguished in which  $P$  may stand to  $Q$ ; (a) Implicative, leading to the *Ponendo Ponens*; (b) Counter-implicative, leading to the *Tollendo Tollens*; (c) Alternative, leading to the *Tollendo Ponens*; (d) Disjunctive, leading to the *Ponendo Tollens*:

- (a) If  $P$  then  $Q$ , but  $P$ ; therefore  $Q$ .
- (b) If  $Q$  then  $P$ , but not  $P$ ; therefore not  $Q$ .
- (c) Either  $P$  or  $Q$ , but not  $P$ ; therefore  $Q$ .
- (d) Not both  $P$  and  $Q$ , but  $P$ ; therefore not  $Q$ .

In these composite premisses, we shall take the implicates and alternants to stand for universal propositions, and the implicants and disjuncts to stand for particular propositions. This secures, for each case, a form of inference in which a particular or singular premiss yields a universal conclusion. Thus:

- (a) If some  $S$  is  $p$ , then every  $T$  is  $q$ ;  
but this  $S$  is  $p$ ,  
∴ every  $T$  is  $q$ .
- (b) If some  $T$  is  $q$ , then every  $S$  is  $p$ ;  
but this  $S$  is  $p'$ ,  
∴ no  $T$  is  $q$ .
- (c) Either every  $S$  is  $p$ , or every  $T$  is  $q$ ;  
but this  $S$  is  $p'$ ,  
∴ every  $T$  is  $q$ .
- (d) It cannot be that some  $S$  is  $p$ , and some  $T$  is  $q$ ;  
but this  $S$  is  $p$ ,  
∴ no  $T$  is  $q$ .

In the above formulae, it will be observed that the simple or categorical premiss is not the precise equivalent or contradictory, as the case may be, of the corresponding proposition that occurs in the composite premiss; for 'this  $S$  is  $p$ ' is more determinate than



'some  $S$  is  $p$ ,' being one of its superimplicants; and again 'this  $S$  is  $p$ ' is not the mere contradictory of 'every  $S$  is  $p$ ,' being one of its contraries or superopponents. The categorical premiss having been in this way strengthened, the conditions of valid inference are still satisfied. In short, we have taken as our instantial premiss a specific instance characterised determinately. The object of this is to illustrate the symbolic formulae by concrete examples which, when further developed, will exhibit the nature of demonstrative induction in its most important scientific forms. Consider the following illustrations of the symbolic formulae:

(a) If some one recorded miracle has been shown to have happened, then every natural phenomenon has a supernatural factor; but such or such recorded miracle has been shown to have happened; therefore every natural phenomenon has a supernatural factor.

(b) If some one female member of a Board had lowered the educational standard in her university, every woman would have submitted to exclusion from the Cambridge Senate; but Miss C. has not submitted to exclusion from the Cambridge Senate; therefore no female member of a Board has lowered the educational standard in her university.

(c) Either every Protectionist country is financially handicapped or every economist of the old school is mistaken; but America is commercially prosperous; therefore every economist of the old school was mistaken.

(d) It cannot be that some variations can be artificially produced in domesticated animals, while there are some species whose characters are unaffected by their environment; but some variations have been artificially produced in the pigeon; therefore there are no

species whose characters are unaffected by their environment.

These illustrations would be regarded by those logicians who divide *all* inferences into inductive and deductive, as being of the nature of deduction rather than of induction, because the universal conclusion is not a generalisation of the instantial premiss. In contrast to these we will therefore now select a set which will be recognised as of the nature of induction; inasmuch as here the universal conclusion in each case *is* a generalisation of the instantial premiss. These new examples are applications of the same symbolic formulae as the preceding set; they differ only in that the symbols  $S$  and  $T$  will now stand for the same class, whereas in the first set they stood for different classes.

(a) If some boy in the school sends up a good answer, then all the boys will have been well taught; the boy Smith has sent up a good answer; therefore all the boys have been well taught.

(b) If a single authoritative person had witnessed the alleged occurrence, then everyone would have believed it; but Mr S. is incredulous; therefore no authoritative person could have witnessed the occurrence.

(c) Either every act of volition is determined or every act of volition is free; but by introspection I am sure that a certain act of mine was undetermined; therefore every volition is free.

(d) It is impossible to suppose that any modern theologians are genuine scholars while others have remained orthodox; Dean I. is a genuine scholar; therefore no modern theologian could have remained orthodox.

§ 3. Returning to the symbolically expressed formulae, and substituting  $p$  or  $\bar{p}$ , as the case may be, for  $q$ , as well as  $S$  for  $T$ , the composite premisses will assume the following still more specialised form:

- (a) If some  $S$  is  $p$  then every  $S$  is  $p$ .
- (b) If some  $S$  is  $\bar{p}$  then every  $S$  is  $\bar{p}$ .
- (c) Either every  $S$  is  $p$  or every  $S$  is  $\bar{p}$ .
- (d) Not both some  $S$  is  $p$  and some  $S$  is  $\bar{p}$ .

It will be seen that these four composite premisses are formally equivalent to one another, and that by adding the categorical premiss 'This  $S$  is  $p$ ' we may conclude in each case that 'Every  $S$  is  $p$ .' Now we may transform the alternation of universals in (c) and the disjunction of particulars in (d) by substituting for  $p$  and  $\bar{p}$  any set of predicates  $p, q, r, t, v \dots$  for the alternative proposition (c), and the same set in pairs for the disjunctive proposition (d), thus:

(c) Either every  $S$  is  $p$  or every  $S$  is  $q$  or every  $S$  is  $r$  or ... etc.

(d) Not both 'some  $S$  is  $p$  and some  $S$  is  $q$ ' and not both 'some  $S$  is  $p$  and some  $S$  is  $r$ ' ... etc. etc.

In this transformation the two complexes (c) and (d) are no longer equivalents but rather complementaries to one another. If the categorical premiss 'This  $S$  is  $p$ ' is now introduced we may infer by means of (d) that 'No  $S$  is  $q$ ,' 'No  $S$  is  $r$ ,' 'No  $S$  is  $t$ ' etc., so that all but the first of the universal alternants in (c) is rejected, and again the universal conclusion 'Every  $S$  is  $p$ ' is established. The need of combining the complementaries (c) and (d) in order to establish the required universal conclusion is apparent when we consider a con-

crete illustration. In the following example, where the predicates  $p, q, r \dots$  stand respectively for 'attacking the Coalitionists,' 'attacking the Liberals,' 'attacking the Labour Party' ... it will be observed that the composites (c) and (d) retain the same logical force as in the above symbolisation, although somewhat differently worded:

(c) At least one of the political parties was attacked by every speaker at a certain sitting of the Congress, and (d) not more than one of the parties was attacked at that sitting.

Mr X. who spoke attacked the Coalitionist Party.

∴ Every speaker at the sitting attacked the Coalitionist Party.

§ 4. Now, if—instead of  $p, q, r \dots$ —we take determinates  $p, p', p'' \dots$  under the same determinable  $P$ , then the disjunctive premiss (d) will not be explicitly required, because it is accepted *a priori* that nothing can be characterised by both of any two determinates under the same determinable. What remains then is the universal alternative proposition (c), established, we may assume, by problematic induction; namely: 'Either every  $S$  is  $p$ , or every  $S$  is  $p'$ , or every  $S$  is  $p'' \dots$  running through all the determinates under  $P$ ,' and this may be summed up in the single phrase 'Every  $S$  is characterised by *some the same* determinate under the determinable  $P$ .' If to this composite premiss is added the instantial premiss 'This  $S$  is  $p$ ,' the universal conclusion follows that 'Every  $S$  is  $p$ .' This trio of propositions represents the one immediate way of establishing a generalisation demonstratively from a single instance, and it will be termed

*The Formula of Direct Universalisation*

Composite Premiss: Every *S* is characterised by some the same determinate under the determinable *P*.

Instantial Premiss: This *S* is *p*.

Conclusion:  $\therefore$  Every *S* is *p*.

§ 5. To take a typical illustration from science:

Every specimen of argon has some the same atomic weight.

This specimen of argon has atomic weight 39.9.

$\therefore$  Every specimen of argon has atomic weight 39.9.

In this, as in all such cases of scientific demonstration, the major premiss is established—not *directly*, by mere enumeration of instances—but rather by deductive application of a wider generalisation which has been ultimately so established. In the given example it is assumed that *all* the chemical properties of a substance, defined by certain 'test' properties, will be the same for all specimens; and this general formula is applied here to the specific substance *argon*, and to the specific property *atomic weight*. The assumption in this case is established by problematic induction, i.e. directly from an accumulation of instances. In practically all experimental work, a single instance is sufficient to establish a universal proposition: when instances are multiplied it is for the purpose of eliminating errors of measurement. It is owing to the fact that the general proposition, functioning as major or supreme premiss, has the special form of an alternation of universals that, by means of a minor premiss expressing the result of a single observation, we are enabled to establish a universal conclusion. This conclusion, in accordance with

our general account of demonstrative induction, is a specification of what is predicated indeterminately in the universal premiss, and a generalisation of the proposition recording the result of a single observed instance.

§ 6. The most important extension of demonstrative induction deals with such methods as those of agreement and difference that have been treated by Mill. We propose to give a formal account of methods similar to those explained by Mill, but so constructed as to render them strictly demonstrative. Many critics of Mill's methods have treated them disparagingly because of his failure to exhibit their formal cogency; while others have maintained that induction should not profess to exhibit the strictly formal character that is ascribed to syllogism and other deductive processes. I hold, on the contrary, that Mill's methods can and should be exhibited as strictly formal, by rendering explicit certain implicit premisses upon which the cogency of the argument from instances in any given case depends; and by indicating the precise conclusion which can be drawn from the instances in question. The implicit premiss is ultimately established by a process of problematic induction, which must be sharply distinguished from the demonstrative process exemplified by the methods. Mill's exposition differs from mine, then, in three preliminary respects. In the first place, he does not clearly distinguish the nature of direct or problematic induction from the nature of the process conducted in accordance with his 'methods of induction,' which he appears often to regard as demonstrative. This confusion is particularly noticeable when we contrast his different modes

of treating the methods of Agreement and of Difference: 'Agreement' he hardly distinguishes from the method of simple enumeration, which is admittedly problematic; whereas 'Difference' he attempts to exhibit as strictly demonstrative. In the second place, he professes to employ as the 'supreme major premiss' for his methods a very wide but at the same time undefined proposition called the 'Law of Causation.' In opposition to this prevalent view, I hold that it is impossible to present such methods as those of Agreement and Difference as strictly formal so long as we attempt to subsume them under so vague a proposition as the Law of Causation, and that each inference drawn in accordance with these methods requires *its own* specific major premiss. The formulation of such a major premiss is the necessary first step in rendering formally cogent any inference (drawn under methods similar to Mill's) from instances finite in number, presented either in passive observation or under experimental conditions. In the third place, whereas Mill retains or eliminates a determining factor according as it affects or does not affect a determined character, in my view the precise conclusion to be drawn is not correctly expressed in terms of the presence or absence of factors, but rather in terms of co-variation, thus: according as in two instances a single variation in any determining character does or does not yield a variation in the determined character, the same will hold for any and every further variation of that determining character.

§ 7. In order to obtain the requisite premisses for demonstrative induction, we must assume that by a preliminary inductive process based upon general ex-

perience, a number of variable circumstances have been eliminated as irrelevant to the formula to be proved. The exposition of this preliminary process by which irrelevant conditions are eliminated, must be postponed until we examine in detail the nature of problematic induction. The process itself must be regarded as pre-scientific; and science takes up the problem at the point where the character of a phenomenon is known to depend only upon a limited number of variable conditions. This knowledge is expressed in a proposition which constitutes the major premiss in the scientific process which we are about to examine as a species of demonstrative induction. The major in question is specifically different for different classes of phenomena, and is in this respect unlike the so-called Law of Causation which professes to be the same for every class of phenomena. If the symbols  $A, B, C, D, E$  are taken to illustrate the determining characters, and  $P$  the thereby determined character, then the instances collected in order to establish a given generalisation of the form  $ABCDE \sim P$ , must be characterised by the same set of determinables, and will be said to be of the same type or homogeneous with one another. The specific major premiss may then be expressed in the formula:

The variations of the phenomenal character  $P$  depend *only* upon variations in the characters  $A, B, C, D, E$  (say).

The conception of dependence, which the above formula introduces, requires more precise explanation. In the first place the formula must be understood to imply that the variations of  $A, B, C, D, E$ , upon which

variations of  $P$  depend, are independent of one another. For if, for example, a variation of  $A$  entailed a variation of  $B$ , then  $B$  being a *determined* character should be omitted from amongst the *determining* characters. It is only by observing this principle that we can apply the essential rule for all experimentation—that *one* only of the determining characters should be varied at a time. Again it is essential that  $A, B, C, D, E$ , should be simplex characters: for the nature of the dependence of  $P$  upon them is such that, if only one of these mutually independent determining characters varies, the character  $P$  will vary; whereas, if more than one of them varied,  $P$  might remain constant. This consideration shows that if any character such as  $A$  was not simplex, but resolvable into unknown factors  $X$  and  $Y$  which varied independently of one another, then a variation in  $A$  might involve such a variation in both  $X$  and  $Y$  that the character  $P$  would remain unchanged.

In the second place, the force of the term 'only' indicates that the dependence of  $P$  upon  $A, B, C, D, E$  is such that no variable circumstances other than these need be taken into consideration, all others having been previously eliminated in what we have called the prescientific or problematic stage of the induction. The conclusion that results from this prescientific induction is to be expressed by an alternation of universals in the form: 'Either every instance of  $abcde$  is  $p$ , or every instance of  $abcde$  is  $p'$ , or every instance of  $abcde$  is  $p''$ , or ....' From this it follows that when a single instance is given of  $abcde$  that is  $p$ , this may be immediately universalised in the form 'Every  $abcde$  is  $p$ .'

It should be pointed out that this immediate universalisation is not dependent upon any comparison of one instance with another, and is prior to the use of such methods as those of difference or agreement; being in fact exemplified above for the case of the atomic weight of argon.

The full significance of the notion of dependence is brought out by taking not only instances which agree in the determining characters and therefore in the determined character, but by taking also instances which differ in the determining, and consequently also in the determined characters. If a variation in any one of the characters  $A, B, C, D, E$  entails a variation in  $P$ , then, in accordance with the principle underlying Mill's method of Difference, that character cannot be eliminated; whereas, if no variation in  $P$  is entailed by a variation in some one of the characters  $A, B, C, D, E$ , then, in accordance with the principle underlying Mill's method of Agreement, that character *can* be eliminated.

§ 8. The forms of Demonstrative Induction to be now exhibited contain (1) the supreme premiss of dependence formulated above for a given set of determinables, and (2) a finite set of instancial premisses under the same determinables. These forms will be distinguished under four heads to be designated *figures* rather than *methods*; but will not correspond severally to Mill's methods, although primarily based upon his method of Difference, and with some important modifications upon his method of Agreement. The notion of 'figure' is substituted for that of 'method'; (a) because there is only one method employed in the four figures, namely that of varying one determining factor



at a time; and (b) because, as in the case of the figures of syllogism, the precise conclusion drawn from the instantial premisses will depend on the nature of the instances themselves, and the figure to be employed in any given case will not be foreknown until the instances have been examined and compared. I shall adopt the phrases 'Difference' and 'Agreement' for the first two figures but 'Composition' and 'Resolution' for the two remaining figures. All the four figures have the same demonstrative force, and the two last figures—though they have some resemblance to Mill's or rather Herschel's method of Residues, which, as shown in a previous chapter, is purely deductive—have precisely the same inductive nature as those of Difference and Agreement. In each figure, the first step in the demonstrative process is to universalise each single instance *taken separately* in accordance with the principle of Direct Universalisation enunciated above; and the second to draw the more specific conclusion that can be inferred from a *comparison* of instances.

We proceed to give an account of each of the four figures in turn.

### § 9. *Figure of Difference*

Given the supreme premiss:  $P$  depends only upon  $ABCDE$ : we shall suppose instantial premisses in which variations occur in the determining factor  $D$ , which is assumed to be simplex.

Then a single instance of  $abcde$  that is  $p$  is universalised into 'Every instance of  $abcde$  is  $p$ .'

Again a single instance of  $abcd'e$  that is  $p'$  is universalised into 'Every instance of  $abcd'e$  is  $p'$ .'

Comparing these two instances of  $abce$ , we note that a variation from  $d$  to  $d'$  entails a variation from  $p$  to  $p'$ .

From this we infer that the value of  $D$  is actually operative in determining the value of  $P$ . Hence any further variation of  $D$ —say from  $d$  to  $d''$ —will entail a further variation of  $P$ —say from  $p$  to  $p''$ . I.e. any value of  $D$  other than  $d$  or  $d'$  will yield a value of  $P$  other than  $p$  or  $p'$ .

Represented symbolically, the conclusion reached is that

'Every instance of  $abcd''e$  will be  $p''$ ,'

where this universal is interpreted to signify that, within the range  $abce$ , any given difference in  $D$  will entail *some* difference in  $P$ , without however indicating what determinate value of  $P$  will be yielded by the given determinate value of  $D$ .

We may symbolise the form of inference which has just been explained in the following scheme:

#### *Figure of Difference*

*Supreme Premiss*:  $P$  depends only on  $A, B, C, D, E$  where  $D$  is simplex.

| Instancial Premisses                                      | Immediate Conclusions                    |
|-----------------------------------------------------------|------------------------------------------|
| 1. A certain $abcde$ is $p$ .                             | $\therefore$ 1. Every $abcde$ is $p$ .   |
| 2. A certain $abcd'e$ is $p'$ .                           | $\therefore$ 2. Every $abcd'e$ is $p'$ . |
| Final Conclusion: $\therefore$ Every $abcd''e$ is $p''$ . |                                          |

### § 10. *Figure of Agreement*

Given the supreme premiss:  $P$  depends only upon  $ABCDE$ : we shall suppose instantial premisses in which variations occur in the determining factor  $A$ , which is assumed to be simplex.

Then a single instance of  $abcde$  that is  $p$  is universalised into 'Every instance of  $abcde$  is  $p$ .'

Again a single instance of  $a'b cde$  that is  $p$  is universalised into 'Every instance of  $a'b cde$  is  $p$ .'

Comparing these two instances of  $b cde$ , we note that a variation from  $a$  to  $a'$  entails no variation in  $P$ .

From this we infer that the value of  $A$  is not actually operative in determining the value of  $P$ . Hence any further variation of  $A$ —say from  $a$  to  $a''$ —will entail no variation in  $P$ ; i.e. any value of  $A$  will yield the same value  $p$  of  $P$ .

Represented symbolically, the conclusion reached is that:

'Every instance of  $A b c d e$  will yield  $p$ ,'

where this universal is interpreted to signify that within the range  $b c d e$ , whatever value  $A$  may have, the value  $p$  will remain unaffected.

We may symbolise the form of inference which has just been explained in the following scheme:

#### Figure of Agreement

*Supreme Premiss:*  $P$  depends only on  $A, B, C, D, E$ , where  $A$  is simplex.

| Instantial Premisses                                    | Immediate Conclusions                    |
|---------------------------------------------------------|------------------------------------------|
| 1. A certain $abcde$ is $p$ .                           | $\therefore$ 1. Every $abcde$ is $p$ .   |
| 2. A certain $a'b cde$ is $p$ .                         | $\therefore$ 2. Every $a'b cde$ is $p$ . |
| Final Conclusion: $\therefore$ Every $a'b cde$ is $p$ . |                                          |

#### § 11. Figure of Composition

Given the supreme premiss:  $P$  depends only upon  $ABCDE$ : we shall suppose instancial premisses in which variations occur in the determining factor  $C$ , which is assumed to be simplex.

Then a single instance of  $abcde$  that is  $p$  is universalised into 'Every instance of  $abcde$  is  $p$ .'

Again a single instance of  $abc'de$  that is  $p'$  is universalised into 'Every instance of  $abc'de$  is  $p'$ .'

Comparing these two instances of  $abde$  we could infer, as in the Figure of Difference, that a further variation of  $C$  would entail a variation in  $p$ . But we have to contemplate a third instance where  $c''$  yields the same value  $p$  that was presented in the first instance. If the values  $abe$  are known to be the same as in this first instance, then a difference in the remaining factor  $d$  must have accounted for the recurrence of the same determined value  $p$ . Thus the first and third instances of  $abe$  determining  $p$  must have been due to the compounding of  $c$  with  $d$  in the first case, and to the compounding of  $c''$  with  $d''$  in the third case. Such a case arises when the factor  $D$  in the third instance has not been amenable to precise evaluation.

Represented symbolically the conclusion reached is that:

'Any instance of  $abc''pe$  will be  $d''$ ,'

where  $d''$  is some unevaluated value of  $D$  other than  $d$  or  $d'$ .

Symbolically:

#### Figure of Composition

*Supreme Premiss:*  $P$  depends only on  $A, B, C, D, E$ , where  $C$  is simplex.

| Instantial Premisses                                      | Immediate Conclusions                    |
|-----------------------------------------------------------|------------------------------------------|
| 1. A certain $abcde$ is $p$ .                             | $\therefore$ 1. Every $abcde$ is $p$ .   |
| 2. A certain $abc'de$ is $p'$ .                           | $\therefore$ 2. Every $abc'de$ is $p'$ . |
| Final Conclusion: $\therefore$ Every $abc''pe$ is $d''$ . |                                          |

§ 12. *Figure of Resolution*

Given the supreme premiss:  $P$  depends only upon  $ABCDE$ : we shall suppose instantial premisses in which variations occur in the determining factor  $E$ , which is not here assumed to be simplex.

Then the three single instances of

$$abcde \sim p, \quad abcde' \sim p', \quad abcde'' \sim p,$$

may be respectively universalised into

Every  $abcde$  is  $p$ , Every  $abcde'$  is  $p'$ , Every  $abcde''$  is  $p$ .

Comparing the first and third of these instances, where under the range  $abcd$ ,  $e$  and also  $e''$  yield  $p$ , we conclude that  $E$  is complex, being resolvable say into the two independent factors  $X$ ,  $Y$ ; so that (say)  $e=xy$ , and  $e''=x''y''$ .

Represented symbolically, the conclusion reached is that

'Every  $abcdxy$  is  $p$ , and Every  $abcdx''y''$  is  $p$ ,'

where  $xy$  and  $x''y''$  represent the resolution of  $e$  and  $e''$  to account for the same value  $p$  of  $P$ . Thus:

*Figure of Resolution*

*Supreme Premiss:*  $P$  depends only upon  $A, B, C, D, E$ .

| Instanial Premisses             | Immediate Conclusions       |
|---------------------------------|-----------------------------|
| 1. A certain $abcde$ is $p$ .   | 1. Every $abcde$ is $p$ .   |
| 2. A certain $abcde'$ is $p'$ . | 2. Every $abcde'$ is $p'$ . |
| 3. A certain $abcde''$ is $p$ . | 3. Every $abcde''$ is $p$ . |

Final Conclusion:  $E$  is resolvable into  $XY$ ,  
where  $e=xy$ , and  $e''=x''y''$ .

§ 13. It will be seen that each of these figures of inductive implication is formally equivalent to a single

disjunction of four propositions. This fourfold disjunction may be called:

*The Antilogism of Demonstrative Induction*

Given three instances of the same type exhibiting three different values of a given determining character, then no case can arise in which:

- (1) the *given* determining character is simplex;
- (2) the values of the *other* determining characters agree throughout the three instances;
- (3) the value of the determined character *differs* in two of the three instances;
- (4) the value of the determined character *agrees* in two of the three instances.

Symbolically expressed, we cannot have

$B$  simplex; and

an instance of  $a b c d e$  that is  $p$ ,  
" " "  $a b' c d e$  that is  $p'$ ,  
" " "  $a b'' c d e$  that is  $p$ .

Expressing this fourfold disjunction in terms of its four equivalent implications, we can formulate the Figures of Demonstrative Induction thus:

- (not-4) *Figure of Difference*: If 1 and 2 and 3; then not-4  
(not-3) *Figure of Agreement*: If 1 and 2 and 4; then not-3  
(not-2) *Figure of Composition*: If 1 and 3 and 4; then not-2  
(not-1) *Figure of Resolution*: If 2 and 3 and 4; then not-1.

In symbols this becomes:

*Figure of Difference.*

If  $B$  is simplex, and we have

an instance of  $a b c d e$  that is  $p$ ,  
and an instance of  $a b' c d e$  that is  $p'$ ,  
then every instance of  $a b'' c d e$  will be  $p''$ .

*Figure of Agreement.*

If  $B$  is simplex, and we have

an instance of  $a b c d e$  that is  $p$ ,  
and an instance of  $a b' c d e$  that is  $p$ ,  
then every instance of  $a b'' c d e$  will be  $p$ .

*Figure of Composition.*

If  $B$  is simplex, and we have

an instance of  $a b c d e$  that is  $p$ ,  
and an instance of  $a b' c d e$  that is  $p'$ ,  
then any instance of  $a b'' c p e$  must be  $d''$ .

*Figure of Resolution.*

If we have an instance of  $a b c d e$  that is  $p$ ,  
and an instance of  $a b' c d e$  that is  $p'$ ,  
and an instance of  $a b'' c d e$  that is  $p$ ,  
then  $B$  is complex.

§ 14. A simple illustration of the Figure of Difference is afforded by Guy-Lussac's law which connects variations in the pressure  $p$ , temperature  $t$ , and volume  $v$ , of a specific gas  $g$ . Suppose that in two instances without changing  $g$  and  $p$ , a change of temperature from  $t$  to  $t'$  is found to entail a change of volume from  $v$  to  $v'$ . From this it can be inferred under the Figure of Difference that, with the same gas at the same pressure, any further change of temperature, say from  $t$  to  $t''$ , would entail a further change of volume, say  $v$  to  $v''$ . This experiment does not prove that for any other gas or for any other pressure, a change of temperature would entail a change of volume; nor does it indicate what determinate value of the volume would be entailed by any supposed further change of temperature. It should be observed that the conditions required

for the Method of Difference—namely precise constancy in all but one of the determining factors—is much more easily realisable when dealing with the same body or substance and varying its alterable states than when we pass from one to another body or substance in one of which a character is present and in the other absent. Hence the conditions most favourable for the application of the Figure of Difference are those in which concomitant variations in the determining and determined factors are observed. For Mill, on the other hand, the so-called Method of Concomitant Variations was primarily distinguished from the Method of Difference in that the latter was concerned with presence and absence, and the former with variations in degree. He speaks of this method as the one necessarily required when we cannot wholly get rid of a phenomenon, and are obliged to be satisfied with noting the varying degrees with which it is manifested from instance to instance; as if this method were a sort of makeshift which had to be put up with when recourse to the Method of Difference was impossible. But it is precisely in those cases in which we can vary the degree of a phenomenon, and not in those that can be described as presence and absence, that we can be assured that the rigid conditions required by the Method of Difference are fulfilled. Mill in adopting this position neglected the consideration of the homogeneity in any collection of instances brought together for comparison under any method of induction whatever. In the conception of concomitant variations is included—not only quantitative variations or variations of degree but also qualitative variations under any

given determinable such as colour or sound. To illustrate Concomitant Variations, Mill chose the method employed in connecting the varying heights of the tides with the variations of the position of the sun and moon relatively to the earth; but he presented the matter as if the difference in the cogency of this method from that of Difference was due to the distinction between presence and absence in the latter and variations of degree in the former; whereas it is obvious that the real deficiency in this application of the Method of Concomitant Variations was due to the special nature of the case, which made it impossible to secure, in the different instances examined, exact agreement in regard to the circumstances not known to be irrelevant: e.g. the variations of height of the tides might have depended upon variations in the force or direction of the wind, or in the shape of the coast, etc. So far then from regarding the Method of Concomitant Variations as an inferior substitute for that of Difference, if by the former is meant variation in the alterable states or relations of some one body or substance, and by difference is meant comparison of two similar bodies in one of which some quality is present and in the other absent, we must regard the former method as superior to the latter. For example: if we attempt empirically to establish a causal connection between the prosperity or the reverse of a country and its adoption of free trade or protection, it would be impossible to find two different countries which agreed in all relevant respects with the exception of this difference in industrial policy; and hence a change in which one policy was replaced by the other within one and the same country

would afford incomparably more cogent evidence of causation than a comparison of the effects in two different countries which must necessarily differ in very many respects that could not be assumed to be irrelevant.

§ 15. To illustrate the Figure of Agreement we may take instances used to establish the law that the rate at which a body falls *in vacuo* to the earth is independent of its weight. In these instances we keep unchanged all the possibly relevant circumstances, such as distance from the earth, absence of air, substance and shape of the falling body, and vary only the weight. From two instances in which the weight alone differs, we find that the time occupied in falling through any given distance is unchanged. In this way we use the Figure of Agreement which might also be called the Figure of Indifference, since it picks out a determining condition which is naturally expected actually to modify the effect in question, and yet is shown by a comparison of instances to be *indifferent* as regards the determinate value of the effect. An illustration of this kind seems not to have occurred to Mill, because in his Method of Agreement every circumstance except one *differs* in the several instances; whereas, in my formulation of the corresponding figure, every circumstance except one *agrees* in the several instances. In other words, as regards the determining factors my Figures of Difference and of Agreement require the same condition, namely a single difference; whereas Mill contrasts the two by defining the Method of Difference as involving a single difference and the Method of Agreement as involving a single agreement. In fact Mill attempts



the elimination *en bloc* of all the varying circumstances which distinguish the different instances in which the same effect-value is observed, whereas what is required in order to give corresponding form to the two methods is that we should eliminate as indifferent or irrelevant only one circumstance at a time.

§ 16. Having illustrated the figures of Agreement and Difference, I will explain the strict procedure of using these figures in dealing with a *number* of cause factors and of effect factors conjoined in a set of examined instances. Taking as our original major premiss:  $ABCDE \sim PQRT$ , i.e. the conjunction of the cause factors  $A, B, C, D, E$  determines the conjunction of the effect factors  $P, Q, R, T$ : it is to be remembered that no cause-factors other than those enumerated are determinative of the enumerated effect-factors, as also that no effect-factors other than those enumerated are dependent upon the enumerated cause-factors. We then take in turn one cause-factor and another and find instances from which we may conclude, in regard to a given effect-factor either, in accordance with the Figure of Difference, that the factor that is varied *is* actually operative, or in accordance with the Figure of Agreement, that such factor is *not* actually operative: and this procedure is repeated for each of the effect-characters in turn. Each pair of instances compared in this way will lead to a universal conclusion under the Figure of Difference or of Agreement as the case may be. This *Complete* method (as it may be called) is by no means identical with Mill's Joint Method of Agreement and Difference, the use of which he advocates only to compensate for the failure to

secure variation in a *single* factor; in this Complete method, on the other hand, one cause-factor alone is varied in each pair of compared instances.

This process symbolically expressed serves as an exercise in the application of the principles underlying demonstrative induction. For example, take the following instances:

- (1)  $abcde \sim pqrt$ ; (2)  $d'bcd'e \sim p'q'r't$ ;  
 (3)  $ab'cde \sim p'qr''t$ ; (4)  $a'b'c'de \sim p''q''r't$ ;  
 (5)  $abcd'e \sim p'q'r't$ .

From (1) and (2), we eliminate  $a$  and  $a'$  as irrelevant to  $p$  and  $t$ ; and infer  $Abcde \sim pt$ . From (1) and (3), we eliminate  $b$  and  $b'$  as irrelevant to  $q$  and  $t$ ; and infer  $aBcde \sim qt$ . From (2) and (4), we eliminate  $c$  and  $c'$  as irrelevant to  $r$  and  $t$ ; and infer  $a'bCde \sim r't$ . On the other hand, from the comparison of (1) and (5) we infer that  $d$  and  $d'$  cannot be eliminated as ineffective as regards either  $p, p'$  or  $q, q'$  or  $r, r'$ . Hence, under the Figure of Difference, we infer  $abcd''e \sim p''q''r''$ . Since, however, in these two instances, the variation of  $D$  is inoperative on  $T$ , we also infer, under the Figure of Agreement,  $abcDe \sim t$ . We may now combine the conclusions  $Abcde \sim pt$  and  $abcd''e \sim p''q''r''$ , and thus infer  $Abcd''e \sim p''$ . This conclusion expresses the fact that, under unchanged conditions  $bce$ ,  $A$  is inoperative while  $D$  is operative upon  $P$ . It should be observed that the conclusion  $abcd''e \sim p''q''r''$  is contrary to the inference drawn by Mill in his Method of Difference; for, according to his formulation of the Method, a difference in a single cause-factor entails a difference in a *single* effect-factor. Other inferences—such as

from (1) and (2) that  $a''bcde \sim q''r''$ —may be left to the ingenuity of the reader to discover.

§ 17. Before illustrating the two remaining figures, it is desirable to explain how the symbols employed in my notation are to be practically applied. When the characters of two or more cause-factors are represented by such symbols as  $a, b, c, \dots$  two typical cases may arise; (1) where  $a, b, c, \dots$  represent determinates under different determinables  $A, B, C, \dots$ ; (2) where two or more of them are determinates under the same determinable. In the latter case, supposing the symbol  $B$  to represent the same determinable character as  $A$ , the three factors  $a, b, c$  would more naturally be symbolised by  $a, a, c$ . Here the *recurrence* of the symbol  $a$  indicates that there are two factors conjoined which are *existentially* different the one from the other, although characterisable under the same adjectival determinable. In order to symbolise the two-fold manifestation of cause-factors characterised under the same determinable, we might use the subscripts 1, 2, to represent *existential* plurality; and thus, instead of writing  $ab, ab', a'b, a'b'$ , etc., we should write  $a_1a_2, a_1a_2', a_1'a_2, a_1'a_2'$ , etc. For example, the adjectival determinable *Force* may be represented by  $F$ , and, when two forces enter together as cause-factors in producing a certain effect, the possible variations in which they may be conjoined may be represented by  $f_1f_2, f_1f_2', f_1'f_2, f_1'f_2'$ , etc. Or again: taking the character of a chemical element to be indicated say by its atomic weight, we may use  $A$  to represent this adjectival determinable; and the possible variations in which two elements are conjoined in producing a compound may be represented by  $a_1a_2, a_1a_2', a_1'a_2, a_1'a_2'$ , etc.

Since, however, amongst symbolists, any difference of symbol such as  $a$  and  $b$ , is never understood to prohibit identity of meaning, while of course *an identity of symbol is always understood to prohibit a difference of meaning*, the notation that I have adopted in my schematisation of the Figures may still be retained without danger of confusion; and, in any case, it serves to represent the *general* principles of the Figures, although in specific cases the special notation indicated above may be preferred.

§ 18. As regards the two remaining figures of Composition and Resolution, we must point out their differences from the figures of Agreement and Difference, and explain what is meant by the composition of causes as contrasted with the combination of causes. Although these figures have been exhibited in a form according to which their demonstrative cogency is equivalent to that of Difference or of Agreement, they palpably differ from these latter in two respects. In the first place, the predicate of the universal conclusion drawn in the last two figures concerns one of the *determining* factors  $D$  or  $E$ , while that in the first two figures concerns the *determined* factor  $P$ . In the second place, the last two figures introduce the notion of 'composition' and its converse 'resolution'—these terms being used in a special and technical sense which requires explanation.

§ 19. The notion of composition has been long understood in mathematical physics, where the resultant of two directed forces regarded as components is represented by the diagonal of the parallelogram whose sides represent these components. The principle by

which the mechanical effect of two conjoined forces can thus be calculated, was contrasted by logicians and philosophers with the principle underlying *chemical* formulae, in which the properties of a compound substance could not be calculated in terms of those of the elements combined in the compound. This led to the view that there was a fundamental antithesis between mechanism and chemism, the former of which involved a 'composition,' the latter a 'combination' of cause-factors. Mill introduced and explained the phrase 'composition of causes' (i.e. of cause-factors) and contrasted this with the combination of cause-factors, specially characteristic of chemical phenomena, and also, in his opinion, of many psychological and sociological phenomena. Mill's explanation is not altogether satisfactory. I will therefore attempt my own explanation of the antithesis between composition and combination.

When two cause-factors represented, say, by the determinables  $B$  and  $C$  are such that there are certain pairs of values, say  $bc$  and  $b'c'$ , which jointly determine the *same* value  $p$  of an effect character  $P$ ; then, referentially to  $P$ , the conjunction  $BC$  constitutes a *composition*. On the other hand, when there are no pairs of values under the determinables  $B$  and  $C$ , such as  $bc$  and  $b'c'$ , which jointly determine the *same* value of  $P$ ; then, referentially to  $P$ , the conjunction  $BC$  constitutes a *combination*. What is important to note here is that the distinction between composition and combination is not absolute; for certain conjunctions of cause-factors may constitute a composition referentially to one assigned effect-character, and a combination re-

ferentially to another. For example, when chemical elements are conjoined in producing a compound substance, it is possible to take the weights of certain elements and different weights of other elements so as to produce a compound of the *same* weight; hence referentially to the effect *weight*, the conjunction of chemical elements comes under the principle of composition. But as regards the chemical *character* of the elements conjoined, it is impossible so to vary these as to produce a compound of the same chemical character in two different cases; for instance, the substance having the chemical properties of water can only be produced by the combination of hydrogen and oxygen.

This account of the distinction between composition and combination is to be regarded as an indication rather than as a definition. Expressed mathematically: the conjunction of the factors  $B$  and  $C$  constitute a composition, referentially to the effect  $P$ , when there is a certain function  $f$  such that  $p$  equals  $f(b, c)$  for any and every value  $b$  and  $c$  of  $B$  and  $C$ . We might therefore replace the terms composition and combination respectively by the more suggestive terms *functional* and *non-functional* conjunction. The method of discovering and establishing such functional relations will be treated in the next chapter. But we cannot well illustrate the figures of Composition and Resolution without first modifying their formulation in view of the above explanation of the nature of composition.

§ 20. In the figure of Composition as symbolically formulated, we took two instances agreeing as regards the determining factors  $abde$ , and a third instance agreeing with both as regards  $abe$ , but in which the

factor  $D$  was unamenable to precise calculation. We then supposed that, while in the first two instances the differences  $c$  and  $c'$  in the determining factor  $C$  yielded a corresponding difference  $p$  and  $p'$  in the determined factor  $P$ ; yet, in a third instance,  $c''$  yielded the unexpected effect  $p$  equivalent to that yielded in the first. The unexpectedness of this result was thus accounted for either by our inability, in the third instance, to measure the factor  $D$ , or by our error in supposing that its value was still unchanged. Now, instead of illustrating our figure by supposing equivalence as regards  $P$  in the first and third instance—a somewhat artificial assumption—let us suppose rather that in the third instance the effect, say  $p_s$ , was other than that calculated by a foreknown formula in which the value of  $P$  would be given by  $p'' = f(a, b, c'', e)$ . On the assumption that the correctness of this formula had been properly assured by means of the functional extension of the Figure of Difference or of Concomitant Variations, we should rightly infer that any instance of  $abc''p_s e$  would entail  $d''$  in place of  $d$ , so that the effect  $p_s$ , under the constant conditions  $abe$ , would be due to the composition  $c''d''$ , and not merely to  $c''$ .

In this modified form, the Figure of Composition can be illustrated by the irregular motions from  $p$  to  $p_s$  of the planet Uranus, the positions  $a, b, e$ , of any other planets being effectively unaltered while that of the sun had changed from  $c$  to  $c''$ . The motion from  $d$  to  $d''$  of an unknown planet, afterwards called Neptune, conjoined with that of the sun from  $c$  to  $c''$  accounted for the unexpected movement of Uranus from  $p$  to  $p_s$ ; in other words,  $a, b, e$  being constant,  $p_s$  was the same

function of  $c''$  and  $d''$  as  $p$  was of  $c$  and  $d$ ; so that  $P$  was a function, not of  $C$  alone, but of  $C$  and  $D$  compounded.

A similar illustration of the Figure of Resolution is found in the experiments by which the new chemical substance *argon* was discovered by Sir William Ramsay. Here the factor  $E$  would represent atmospheric nitrogen, and its greater weight—as compared with that of nitrogen prepared from chemical compounds—was accounted for by the *resolution* of the atmospheric nitrogen into the two components argon and pure nitrogen. It should be pointed out that the resolution here employed was not a chemical analysis, for argon does not combine with any other element (as far as is at present known) and therefore the resolution in question was a true instance of the converse of composition.

In regard to the illustration of *Composition* involving the discovery of Neptune, and that of *Resolution* involving the discovery of argon, the precise measurements finally made reduced the inference to a purely deductive form, which assumed the character of the method of Residues according to my interpretation of this method (see p. 118).

## CHAPTER XI

### THE FUNCTIONAL EXTENSION OF DEMONSTRATIVE INDUCTION

§ 1. IN concluding the treatment of demonstrative inference I propose to recapitulate the results that have been so far reached, and to bring into focus the distinctions and connections between the several forms of inference, deductive, inductive and problematic. I have already examined the general notion of function, and shown how it is employed in mathematical and other processes of deductive inference; and it remains to exhibit this notion as it enters into inductive inference—this constituting the specifically new topic to be discussed in the present chapter.

Pure induction, by which is to be understood that which involves no assumption of universal laws, has been shown to be the sole direct and ultimate mode of generalising from instances examined and theoretically enumerable. This species of induction I have called problematic because, in my view, the universal propositions which it establishes must be regarded, not as absolutely certified, but as accepted only with a higher or lower degree of probability depending upon the collective character of the instances enumerated. The possibility of establishing such direct generalisations depends upon certain postulates, the discussion of which raises one of the most important and difficult problems of philosophical logic; and even then, the probability

to be attached to generalisations thus established has to be determined by reference to the formal principles of probability. But, so far as these generalisations enter into the account of demonstration, they function as major premisses. Demonstrative induction, then, so far resembles deduction in that it requires the conjunction of two types of premisses: (1) the major or supreme universal premiss, which expresses the relation of dependence between one specified set of variables and another; and (2) the minor or instancial premiss which sums up the results of single observations or experiments. The major premiss in this mixed form of demonstration is formulated, not as a uniformity pervading all nature, but as a specified universal holding only for the special class of phenomena to which the conclusion refers.

§ 2. A very general statement of the contrast between my exposition and Mill's is conveniently introduced at this point. I have deliberately separated the treatment of formal or demonstrative induction from that of problematic induction. In the latter, the accumulation of instances is all important; in the former, a precise major premiss, relating to a finite and enumerable set of determinables, is required in each step of the formal process. These major premisses are assumed to have been previously established, with a higher or lower degree of probability, on the principles of problematic induction. The essence of problematic as contrasted with formal induction is expressed in three statements: first; no wide generalisation, such as that which asserts the uniformity of nature, is involved; secondly; the instances compared are not



determinately analysed with respect to the variable characters upon which the proposed generalisation may depend; and hence, thirdly, an indefinite multiplication of instances is required in order to give any appreciable value to the probability of the conclusion. It is partly for this reason that Mill's account of the Method of Agreement differs so considerably from my extremely simple Figure of Agreement; for Mill is largely thinking, under the title Agreement, of a direct method of establishing empirical generalisations to which only an inferior degree of probability can be attached. The generalisations thus established by problematic induction function as major premisses in demonstrative processes in one of two ways: either as established with what may be called *experiential* as opposed to *rational* certitude; or as put forward hypothetically, and thus as exhibiting forms of *implication* rather than of inference—implication being defined, as in Chapter I, to be potential or hypothetical inference.

§ 3. The term hypothesis has been used by logicians in so very many senses that, in order to obviate logical confusion, it will be well to examine its various usages, showing how they have developed from one fundamental element. This element will be found to be definitely epistemic rather than constitutive, and for my own purposes I consequently prefer to use the phrase 'hypothetically entertained,' which has an epistemic significance quite independent of the form or content of the proposition so entertained. We may take in turn the various meanings of the substantive 'hypothesis' or the adjective 'hypothetical' that occur

in deductive or inductive logic, in order partly to connect and partly to contrast its epistemic with its other bearings. In traditional formal logic, propositions are called hypothetical which are in fact compounded out of two categorical propositions, say  $p$  and  $q$ . In this case, while the adjective *hypothetical* is traditionally used to denote a particular species of compound proposition, namely that of the form 'if  $p$  then  $q$ '; yet at the same time the term hypothesis clings firstly to the proposition  $p$  because in this form it is not actually asserted, and next to the proposition  $q$  because it is only assertible on condition that  $p$  has been asserted. Thus the adjective *hypothetical* is actually attached to three quite distinct propositions or forms of proposition: the compound 'if  $p$  then  $q$ '; the simple proposition  $p$  itself, which I call the *implicans*; and the simple proposition  $q$  which I call the *implicate*. Now in order to make a first approximation to justifying this confused terminology, we must consider its epistemic aspect, and we may say that normally both the *implicans* separately and the *implicate* separately are entertained hypothetically, while the compound proposition 'if  $p$  then  $q$ ' is entertained assertorically. Hence, even where the term hypothetical is used in its most precise technical sense, it is applied to a form of proposition assumed to be entertained assertorically, the components alone of this assertoric compound being entertained hypothetically.

The recognition of this ambiguity in the use of the term hypothetical resolves the often disputed problem of the relation in general between induction and deduction. When we are concerned with the purely formal

relation of implication as subsisting between the premisses and conclusion of any argument of the general nature of a syllogism, then these premisses need only be entertained hypothetically; while, at the same time, the relation of implication itself is to be conceived, not only as assertorically advanced, but even as having the highest degree or kind of assertoric certitude. The conclusion of a syllogism thus deduced is usually spoken of as demonstrated, i.e. as having demonstrative certitude; although, taken by itself, any kind or degree of certainty attaching to it is wholly dependent upon the kind or degree of certitude with which the premisses are entertained. Taking full advantage, then, of Mill's account of the functions and value of the syllogism, we may say that the hypothetical conclusion has been hypothetically demonstrated, and can only be assertorically demonstrated when we have examined and tested the truth of the premisses. Only when the major premiss has been inductively established can the conclusion be entertained categorically, and even then with a degree of probability dependent upon that of the major premiss; and ultimately upon the mode of induction by which the major has been established.

§ 4. The problematic nature of the universal obtained by induction and functioning as major premiss in a deductive process has led to a confusion between the notions *problematic* and *hypothetical*, resulting in the use of the term 'hypothesis' for any proposition entertained with a degree of probability. Thus, when Jevons says that all induction is hypothetical, what he means is merely that an inductive conclusion has not certainty but probability. Thus any inductive generali-

sation is commonly called a hypothesis; and the term when applied to a scientific theory may have three alternative meanings: first, it may mean that the proposition is unproven; secondly, that the proposition has an appreciable degree of probability which renders it worth considering; thirdly, that the proposition has no appreciable probability at all, and may even be known to be false. Besides the epistemic significance revealed in all these three alternative meanings, the term hypothesis must also be understood to indicate the purpose which an unproven universal, definitely formulated, fulfils in calculating deductively the conclusions to which it would lead. In fact Jevons, in describing induction as hypothetical, uses the term in two quite different senses: first, in the formal sense, to indicate the provisional or tentative attitude towards a universal *before* we have confirmed it by a process involving deduction; and, secondly, to represent the final attitude towards a universal *after* it has been tested and confirmed with the highest attainable degree of probability. With the view indicated in the second application of the term *hypothesis*, I agree; but, as regards the first use of the term, it seems to me that we always adopt a tentative attitude towards a proposition entertained as a proposal, whether it is to be proved deductively or inductively; so that the term as applied to a proposition to be proved does not represent any characteristic peculiar to induction. Now the special topic with which this chapter is concerned involves both the contrasted ideas of hypothesis: namely, of a proposition having a certain degree of probability, and of one put forward to be tested by appropriate evi-

dence. Thus, while the functional formula in deduction is assumed to be true and therefore may serve as premiss for deducing an equally assured conclusion, the inductive aspect of such a functional formula presents the *inverse* problem; for we have now to examine by what kind of instances, and by what modes of comparison, the functional formula itself can be established. So far as this process of examination may be said to have a special characteristic by which it may be distinguished from problematic induction used for establishing the wide generalisations of science, its peculiarity is that a comparatively small number of instances will constitute the sufficient factual basis for the establishment of the formula, and that the actual procedure of mathematical physics, at least in the majority of cases, rightly attaches practical certitude to the formula thus inferred.

§ 5. In order to show how the functional formula is established, I must refer to my account of the figures of Demonstrative Induction. There the conclusion demonstratively drawn does not assign the specific value of the effect-character that is to be correlated with any given value of a cause-character. In popular language, the conclusions drawn would be termed qualitative not quantitative; that is to say, the figures establish causal *connection* without determination of a causal *law* or formula. In comparing the different figures, it is seen that the Figure of Difference, which stands first, is a direct expression of the principle of the dependence of change in the effect upon change in the cause; and that the Figure of Agreement or of Indifference is complementary to that of Difference in the

same sense as the universal or implicative 'if not- $p$  then not- $q$ ' is the complementary of 'if  $p$  then  $q$ '; while the Figures of Composition and Resolution merely carry out the principle of Difference under certain more complicated circumstances. There is, therefore, one principle common to all the four figures, namely that underlying the Figure of Difference—the functional extension of which will be our principal concern.

The original formula of Difference may be restated in the following canon: When in two instances a difference in the cause-character  $D$  entails a difference in the effect-character  $P$ , all other cause-characters which might contribute to the determination of  $P$  being the same in the two instances, then we infer that any other difference in the cause-character will be correlated with some other difference in the effect-character, under the continued constancy of the remaining cause-characters. Now this canon, which applies to two instances only, may be obviously extended to any number of instances all of which conform to the figure of Difference: i.e. all other cause-factors remaining unchanged, we find a series of instances in which  $D$  alone varies, and in which the determinate values  $d, d', d'', d'''$ , etc., say, are associated respectively with  $p, p', p'', p'''$ , etc. Now, as in the simple case of two instances, these observations do not enable us to assign the specific value of  $P$  that is to be correlated with any given value of  $D$ : we can still only infer that any *further* change in  $D$  will be associated with *some* further change in  $P$ . The required extension of the figure of Difference consists, therefore, in the determination of  $P$  as a function of  $D$  which shall hold for all unexamined as well as examined

instances. A famous example of the determination of such a function is that formulated by Kepler who, after nineteen guesses, discovered a formula for the planetary movements about the sun which co-ordinated the spatio-temporal relations for the cases—necessarily finite in number—that he was able to examine and measure. The discovery of this formula involved nothing of the nature of inductive inference, but its application to all the planetary positions *intervening between those observed* constituted a genuine inductive inference, so easy to draw that neither Whewell nor Mill seems to have been aware that any such inference was implicitly made.

The canon for the Figure of Composition may be reformulated as follows: When in several instances variations in the single cause-character  $C$  have entailed variations in the effect-character  $P$  such that, in accordance with the functional extension of the Figure of Difference,  $P$  has been shown to be a certain function of  $C$ , then, if some similar instance of a further variation of  $C$  has entailed a variation of  $P$  *not satisfying this function*, we infer that, in this instance, besides  $C$  some other character, say  $D$ , has varied, and hence that  $P$  depends upon the composition of  $C$  with  $D$ . This simple use of the Figure of Composition does not, however, enable us to determine the value of  $D$  in the particular instance observed. In expanding this figure therefore we have to look for further instances in which both  $C$  and  $D$  can be evaluated; and thus construct a formula by which  $P$  is represented as a function both of  $C$  and of  $D$ . This method should be compared with that of Residues, which I have regarded

as purely deductive; for, in the method of Residues, the values of  $D$  are determined *deductively* from the known formula  $p=f(c, d)$ , whereas, in our extension of the Figure of Composition, the formula  $p=f(c, d)$  is determined *inductively* from the observed values of  $D$ . The case of the irregularities in the movements of Uranus, instanced in the previous chapter, illustrates this type of functional extension.

§ 6. Now the formula which expresses an effect as a function of one or more cause-factors must at least satisfy the negative condition that it fits all the examined instances as regards the observed values of cause and effect. Many logicians, and certainly many experimenters in practical branches of science, are finally satisfied with this negative criterion. They assert, in effect, that provided the formula  $p=f(d)$ , where  $f$  has some specific form, agrees with the values of  $P$  and  $D$  as measured in the examined cases, then it has all the guarantee that experimentation requires for its universalisation. But the mathematician points out that, theoretically speaking, there are an infinity of different functions that would exactly fit any finite number of cases of covariation. Hence he demands in general a much more rigid defence for selecting one formula rather than another to represent the universal law.

In order to escape this threatening annihilation of inductive inference, we may indicate two fundamental principles upon which the highest attainable degree of certainty, which may be called practical or experiential certitude, depends. In the first place, reliance is placed upon the character of the formula itself, and in particular on its comparative simplicity; in the second

place, the higher credibility of a proposed formula depends upon its analogies with other sufficiently well-established formulae in similar classes of phenomena. Briefly, the criteria of simplicity and analogy, especially when conjoined, confer upon a formula of covariation that highest degree of probability which allows us to regard the induction, not as merely problematic, but as virtually demonstrative. For example, the experiments that have been conducted in regard to the covariations of temperature, pressure and volume of gases have always been treated by physicists as conferring absolute demonstrative certitude upon the formulae inferred, although they have been actually confirmed from a necessarily limited number of observations.

We may illustrate the notion of simplicity by taking the simplest of all possible functions, namely where  $p$  is proportional to  $d$ , or its inverse  $\frac{1}{d}$ . For example, if we have instances in which, weight being the determined factor, and some quantitatively measurable cause  $D$  varies so that where we double  $D$  we double  $P$ , and where we treble  $D$  we treble  $P$ , and so on for fractional as well as integral multipliers, we inductively infer that  $P$ , not merely varies *with*  $D$ , but in mathematical language, varies *as*  $D$ . There have been philosophers who, in effect, have imagined that, unless a causal formula can be expressed by a proportionate relation of cause to effect, it must be regarded as a mere empirical rule; and conversely, as soon as instances are found to fit some such simple formula, the generalisation may be regarded as absolutely certified. A slightly less simple kind of formula is exemplified by gravitation

where, for a given attracting mass, the acceleration of the attracted body varies inversely as the square of the distance, being in the direction towards the attracting body. The high probability of this formula is due, not only to its relative simplicity, but to its analogy with the independently known formula for the intensity of radiant light or heat. Moreover the formula in question could have been deduced from the assumption that radiation operates equally in all spatial directions, so that its magnitude upon any part of a spherical surface is inversely proportional to the area of that surface and therefore to the square of the distance. In the examples thus brought forward, indications are given of the kind of reasoning upon which the high probability attached to any formula that fits the examined instances is based.

§ 7. The criterion of simplicity is not often directly applicable; but, when in a relatively complex conjunction of circumstances that can be analysed, a formula is constructed that could have been deduced from a combination of wider and well-established formulae of comparative simplicity, then an empirical formula thus confirmed acquires problematic value corresponding to that of the laws from the combination of which it could have been deduced. Both Whewell and Mill have taken this kind of criterion as fundamental in their theories of induction; Whewell using the phrase 'consilience of inductions,' and Mill having in his earlier chapters put forward this deductive confirmation as the one principle dominating his whole theory. At first sight Mill's position is paradoxical, since he apparently attributes a higher probability-value to a law, merely



on the ground of its width, whereas it would appear that the narrower generalisation is the safer. I think, on this matter, we must recognise the value of the two opposed principles that have been put forward. On the one hand, mere simplicity has been elevated into a supreme criterion; but, so far from admitting that simplicity alone guarantees a formula, we must maintain that where a known complexity of circumstances is involved, a corresponding complexity must be expected to characterise their co-ordinating formula. Hence, when a class of phenomena that have not been definitively analysed resembles other classes for which a complex formula has been established, a corresponding complexity should be anticipated for the given class; whereas the formula for a class of phenomena analogous to others for which a simple formula holds may rightly be expected to be simple. The criterion of simplicity, when including its indirect as well as its direct form, is of value; but it is only when analogy is thus conjoined with simplicity that we may attach practical certitude to a formula which satisfies at least the negative criterion of fitting perhaps only a small number of well-examined cases.

§ 8. The theory of what I have called the functional extension of demonstrative induction constitutes a link between the Demonstrative and the Problematic forms of inference. For certain rules (of a strictly formal character) are required for deducing, amongst all the functions which fit the observed co-variations, the *most probable* function of the variable cause-factors by which an effect-factor may be calculated. The oldest and most usual method of determining this function is known as

the *method of least squares*. Its validity depends upon a certain assumption with regard to the form of the Law of Error, i.e. of the function exhibited by divergences from a mean or average, when the number of co-variational instances is indefinitely increased; and a different method must be employed for each corresponding different assumption. The reader must be referred to Mr J. M. Keynes's *Treatise on Probability*, Chapter XVII, for a very comprehensive and original discussion of this topic.

The inductive inference examined in the above is thus shown to be based upon purely formal and demonstrative principles of probability, whereas the discussion of problematic induction to be developed in Part III will introduce informal theorems of probability, based on postulates of a highly controversial nature. It is therefore legitimate, and even necessary, to include the functional extension of the figures of induction under the general title of demonstrative inference.

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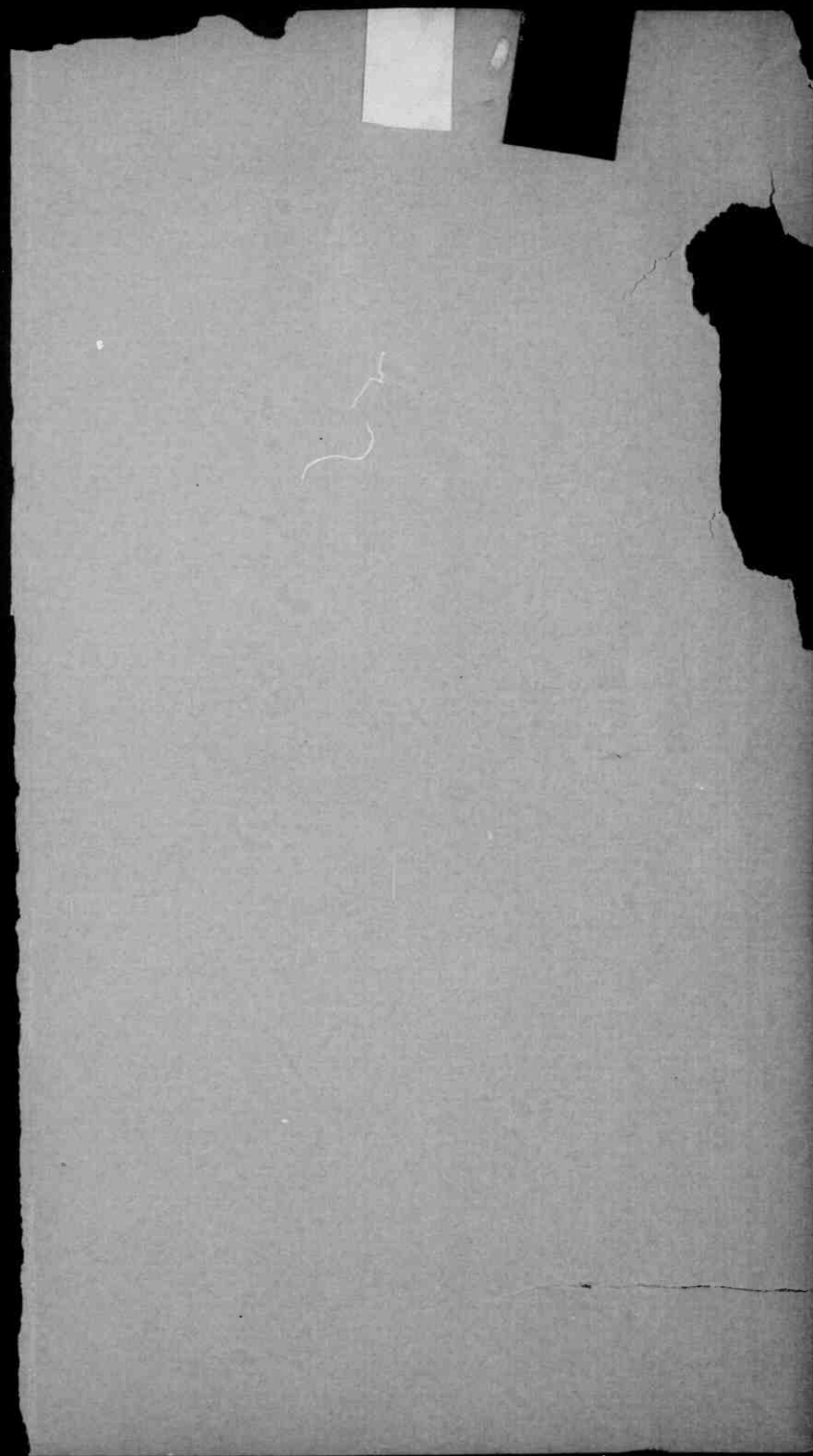
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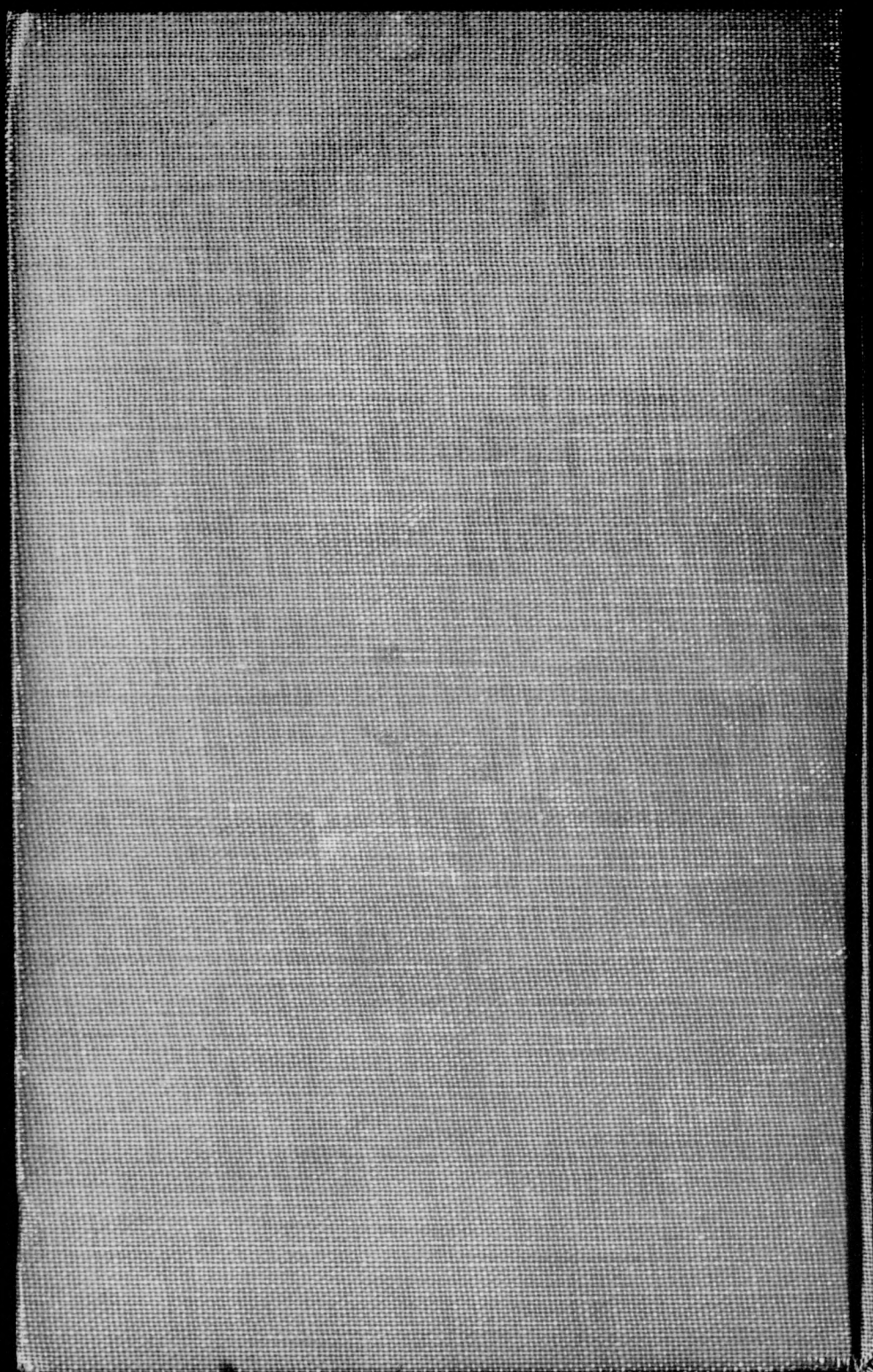
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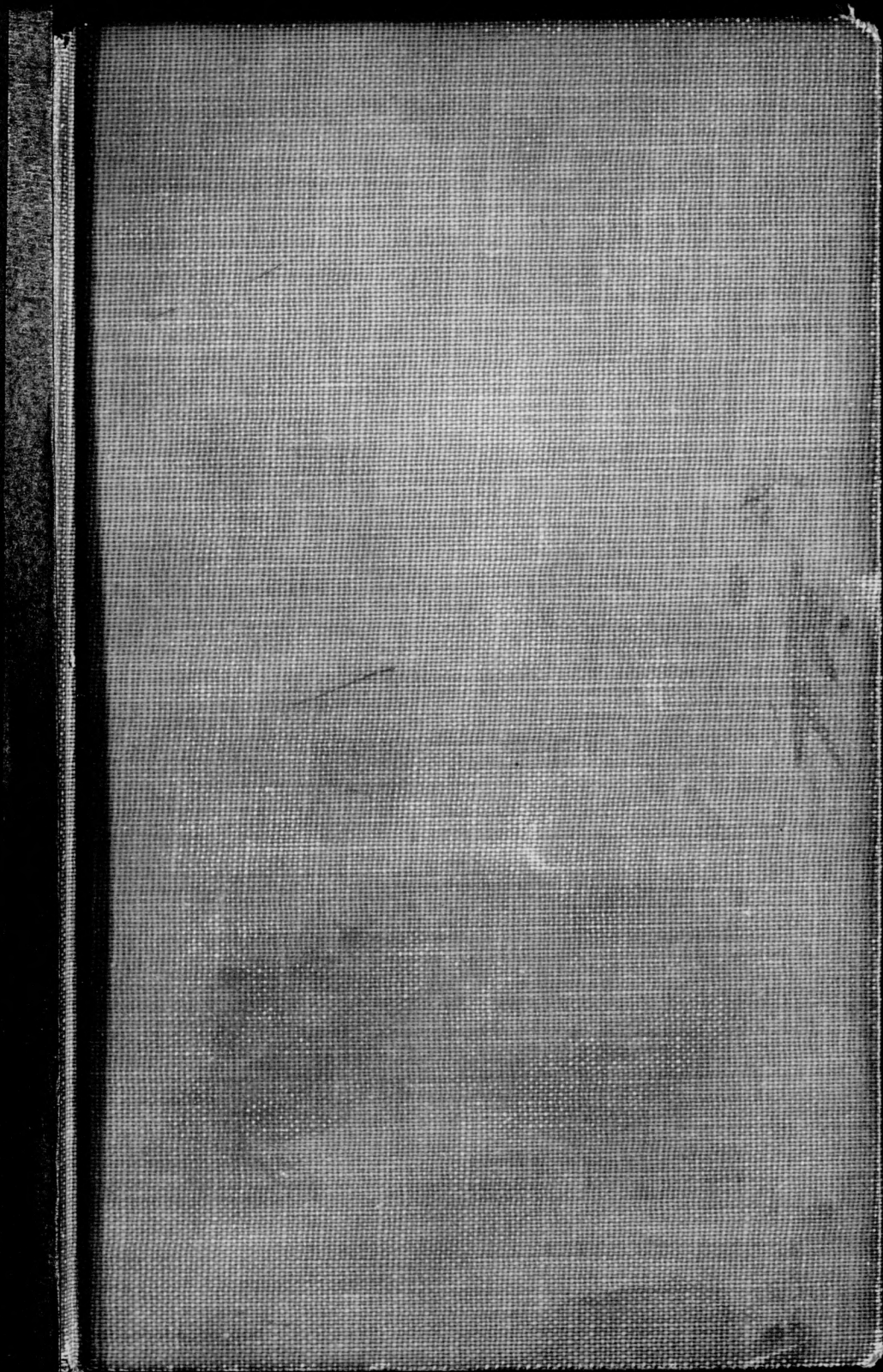
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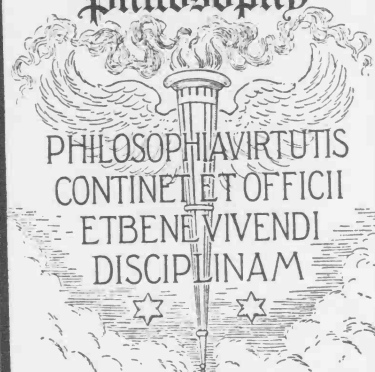


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THE LOGICAL FOUNDATIONS OF SCIENCE

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## INTRODUCTION

§ 1. The subjects discussed in Parts I and II come within the scope of what may be called Formal Logic. Here the proposition is taken to be the immediate object of a possible assertion; and a consideration of its nature leads to the conception of the antithesis and connection of substantive with adjective, as disclosed in the analysis of the simplest articulate form of judgment. The function of language and more particularly of names is examined. It is held that the different forms assumed by compound propositions are indicated by various words, not standing for substantival or adjectival constituents, but expressive of the modes in which simple propositions or their constituents are to be connected by constructive thought. Such considerations lead to a preliminary definition and enumeration of logical categories roughly corresponding to (and replacing) the grammatical enumeration of parts of speech.

In the more detailed examination which follows, substantives proper or existents are distinguished from quasi-substantives, adjectives predicable of the former being termed primary and those predicable of the latter secondary. Modality, in its formal aspects, is treated under the more general heading of secondary propositions. Adjectives are divided into transitive adjectives (otherwise relations) and intransitive adjectives, in precise analogy with the grammatical division of verbs; and again into monadic, dyadic, triadic, etc. according to the number of substantive-terms which are entailed

in their employment. A prominent place is given to the distinction and connection, amongst adjectives in general, between adjectival determinables and adjectival determinates. This distinction is utilised in all the further developments of logical theory. The relations between inference and implication, the former of which is essentially epistemic and the latter essentially constitutive are entered into at considerable length. In particular, certain general and fundamental *principles* of inference are laid down and contrasted as formal with the *premisses* of inference which are material.

Inferences and implications are divided into the two species demonstrative and problematic. The term induction has been used, with some hesitation, to include four species—intuitive, summary, demonstrative and problematic. The first three of these are examined in Part II, the fourth being reserved for Part III. Deductive inference or implication is treated in connection with the intuitive foundations of pure logic and pure mathematics; as also with summary induction.

§ 2. It is contended, in agreement with most recent logicians, that Arithmetic and (more generally) Pure Mathematics develops from Pure or Formal Logic: i.e. that the conceptions and axioms underlying the former are none other than those underlying the latter. If any distinction is to be made between Pre-mathematical Logic and Pure Mathematics it is suggested that the latter introduces certain adjectives and relations which in the strictest sense are constant, i.e. represented by words or symbols of which it is essential for the science that the meanings should be understood in one invariable sense; whereas the intelligent apprehension of

pre-mathematical formulae requires that symbols for adjectives and relations in general should be understood merely illustratively to stand indifferently for any actual adjectives that might be substituted for them.

Now, in the transition from pre-mathematical to mathematical logic, the first notions that demand explicit recognition are those of *identity* and (its contrary) *otherness* or diversity. These two relations are applicable to any entities whatsoever coming under any category whatever. Thus if *a* unambiguously denotes any entity whatever and *b* unambiguously denotes any entity whatever, then (so far) the entity denoted by *a* may be identical with and may be other than that denoted by *b*. At this point, the two axioms that identity and otherness are co-alternate and co-disjunct have to be explicitly formulated. Speaking loosely, the relation of identity yields the notion of *one* and that of otherness yields the notion of *two*. More accurately and precisely the conception of *number* is developed from that of a certain sub-division of the genus *relation* termed *one-one*; and *one-one* relations are defined entirely in terms of identity and otherness; i.e. no other notions than these are involved beyond those appertaining to pre-mathematical logic. In this way, the definition, not only of any assigned finite number, but even of infinite number introduces (besides pre-mathematical notions) *identity* and *otherness* alone. In the higher branches of arithmetic other relations, dyadic, triadic, etc., are introduced, especially those which develop from the general notion of *order*; and these are all expressed and defined in terms of words or symbols having a fixed invariable meaning that must be understood by the mathematician as such.

Not only must the mathematician understand the meanings of the constant symbols introduced and defined in the science, but also his intelligent assent is required to be given to certain axioms (or primarily fundamental propositions) expressed in terms of these symbols; and his intelligence must be further exercised in following the demonstrative procedure by which derivative formulae are progressively inferred. He discovers, not only the comparatively unimportant fact that the conclusions are true provided that the originally premised axioms are true, but also the more important fact that the conclusions follow demonstratively from a judicious combination of these axioms and *these alone*—none other being required. The account of symbolism and allied topics in Part II includes references to processes of thought and thus is largely psychological—in this respect differing from the accounts given by professedly formal logicians.

§ 3. Part III opens new ground. Such ontological conceptions as those of substance and causality—even of 'matter' and 'mind'—are explicitly introduced and their significance discussed in detail. In this way, a claim is made that logic should be recognised as a department of philosophy in a higher sense than any warranted by the restriction of its scope to what has been termed formal logic. It is true that inductive logicians have bestowed much care upon the examination of the nature of cause and, less explicitly, of substance. But for the most part they have deliberately excluded any discussion of the philosophical implications attached to these notions; either on the ground that these implications belong to metaphysics or that they

are to be rejected *in toto* as merely bad metaphysics. For example, though much of what Mill has said and Venn has said better about causal and other uniformities has its value, yet it is obvious that their treatment gives us no instruction on the philosophical questions at issue. Moreover, not only the professedly philosophical logicians but, strangely enough, also the humbler inductive logicians have overlooked or devoted insufficient attention to many methodological problems the discussion of which belongs to the logic of the sciences. This constitutes my apology for entering with considerable detail into topics which lie on the borderland between Logic as Methodology and Logic as Philosophy.

The inductive logicians may be said to have presented a philosophical case only on the supposition that they are to be interpreted as having contended for the inutility of such notions as those of causality and substance in the establishment of scientific generalisations. Thus Mill's reduction of the causal relation to invariable and unconditional sequence is naturally interpreted as tantamount to the rejection of the notion of cause in any philosophical sense. And this is certainly the contention of those among later empiricists who have concerned themselves with the problems of scientific induction. In fact, the more modern view expressly held by formal logicians of the present day (who are mostly empiricists of the school of Hume) is that all the principles of induction (with the doubtful exception of probability) are derivable by an extension of the principles of deduction much as Pure Mathematics is a mere extension of Pure Logic. With this view I am in partial agreement, and the discussions of Part III are largely concerned with

the points both of agreement and disagreement between my view and that of the more extreme empiricists.

In examining the logical foundations of science, I have found it impossible to separate the Epistemological (or preferably Epistemic) from the Ontological point of view. The explanation of this impossibility is that, as it appears to me, certain notions—and certain propositions expressible in terms of these notions—must be *postulated*, if science is to be validly established.

By a postulate I understand a proposition that is assertorically and not merely hypothetically entertained; but yet is adopted neither on the ground of intuitive self-evidence nor of inductive confirmation. More positively, a postulate is framed in terms not given in experience, and these terms enter even into the instancial propositions which are problematically universalised by induction. Postulates, in my view, enter even into mere observations of instances which may otherwise be termed judgments of perception. In these judgments the thinker predicates not merely a concomitance of characters presented to him; but, besides concomitance, causality; and, besides presentment, reference to substance.

§ 4. The ontological discussions of Part III are centred upon the recognition of the two concepts, causality and substance. But I have discarded the term 'substance,' for reasons which need no enumeration, in favour of the term 'continuant.' The genus 'substantive proper,' otherwise termed 'existent,' is divided into the two species 'Continuant' and 'Occurrent.' The distinction among substantives between continuants and occurrents plays a similarly prominent part in material

logic as is played in formal logic by the distinction among adjectives between determinables and determinates. But no *analogy* can be drawn between the antithesis or connection in the one case and that in the other. Negatively, it may be said that a continuant is not a mere collection of occurrents, just as a determinable is not a mere collection of determinates. Further than this we can only say that a plurality of occurrents is constructed by thought into a unity by virtue of the nexus of causality and a plurality of determinates by virtue of the relation of opponency or incompatibility. No *positive* analogy can be drawn, owing (it would seem) to the ultimately irresolvable antithesis between substantive and adjective.

§ 5. A more detailed summary of the views propounded in Part III on ontological problems may now be given.

In the first place, I have adopted the dualistic position which recognises a fundamental distinction between the psychical and the physical, and attributes reality to both in the same unequivocal sense. Whether or not the view is philosophically tenable, at any rate any examination into the principles of science would seem to be impossible without some such hypothesis as that of dualism. Spinoza's acceptance of two unsynthesised attributes,—thought and extension—illustrates, in more or less veiled guise, the very same fundamental position as that adopted by the dualist. But the view that I wish to put forward is less dualistic than Spinoza's, in that I profess to present the psychical and the physical in some sort of synthesis with one another, and not in mere unreconciled antithesis. What I hold to



be important in the dualistic position is the recognition of two kinds of *agency*—psychical agency and physical agency. Of my views, on this and kindred matters, I do not profess to be able to offer any direct demonstration, nor do I believe that my philosophical opponents can offer any valid refutation. The more detailed exposition of my philosophy must be allowed to be taken as a substitute for strict demonstration.

A continuant is defined to be that which continues to exist throughout some limited or unlimited period of time, during which its inner states or its outer connections with other continuants may be altering or may be continuing unaltered. In the first place, then, the continuant must be contrasted with its states—the possessive pronoun here pointing to a unique species of ‘tie’ indicated by the preposition *of* to be understood in a specific sense differing from all other senses. There is no relational word (as far as I know) that can be used to express this specific meaning of ‘of,’ parallel to the relational word *characterising* which expresses the specific meaning of ‘of’ in such a phrase as “the quality *of* this or that.” In fact, the two meanings of the word are continually combined in constructions such as those expressed by the phrase “the quality *of* this or that state *of* this or that continuant.” Just as a quality must be attached or referred to this or that state, so a state must be attached or referred to this or that continuant. We may also speak of a property *of* this or that continuant to mean a property *characterising* this or that continuant, so that *property* (in this application) is a species of the genus *adjective*.

Now while we cannot say that a *continuant* occurs,

we *can* say that a *state* occurs; and anything that may be said to occur will be called an ‘occurrent.’ And I lay it down that any occurrent must be referred to a continuant or to two or more connected continuants. The reference of an occurrent to connected continuants will be entailed when we speak of *transeunt* causality; while the reference of an occurrent to a single continuant will be entailed sometimes in speaking of *immanent* causality and sometimes in speaking of *transeunt* causality.

§ 6. In many applications ‘occurrent’ and ‘event’ may be taken as synonyms; but, properly speaking, they must be distinguished. Thus what is called a *single* event is (or may be) resolvable into a plurality of occurrents of different kinds. The resolution of an event into a plurality of occurrents must not be confounded with the partition of an event into a plurality of parts. The parts of an event are themselves events; and these are distinguished from one another by their difference of spatio-temporal location. On the other hand, the occurrents composing an event cannot be distinguished by difference of location, for they must be located within the same spatio-temporal boundaries as the event itself.

The above general account of the distinction between occurrents and events may be considered first in regard to physical and next in regard to psychical events. A physical event has a spatio-temporal extension which is defined by the spatio-temporal boundary within which it falls, which again determines the four-dimensional magnitude of the extension. In order to distinguish between one and another physical event it would seem, therefore, both necessary and sufficient that we should

be able to assign different spatio-temporal boundaries to the two. This holds even of the event-parts of a whole event as distinguished from one another and from the whole; the different event-parts being said to occupy different parts of the extension occupied by the whole event. Now, besides mentally dividing an event into parts, we may also mentally resolve an event into occurrents. The several occurrents which thus compose an event are distinguished, not by the spatio-temporal position which they occupy, but by the different adjectival determinables under which their determinate characters fall. Now all that is here said about physical events and physical occurrents holds also of psychical events and psychical occurrents, except for the fact that spatial reference cannot be applied to the latter and temporal reference only remains. It follows that the extension of a psychical event and the magnitude of its extension are one-dimensional instead of four-dimensional. Hence, whereas difference of position would seem to be necessary and sufficient to mark off one physical event from another, difference of dating is not necessary or sufficient for marking off one psychical event from another. Thus, if one person is suffering tooth-ache contemporaneously with another person's reflecting upon a mathematical problem, we should speak of these as *two* events, although we cannot attribute to either of them spatial extension or boundary and, therefore, cannot attribute to them *different* spatial extensions or boundaries.

This shows that in order mentally to separate one psychical event from another we must postulate, not only a difference of temporal position (if any), but also

different psychical continuants to which the two different psychical events are to be referred. *A priori*, indeed, the same must hold as regards physical events; i.e. two simultaneous events might occupy the same locality, which is tantamount to the possibility that two bodies (physical continuants) should be 'occupying' the same place at the same time. This postulate would be necessitated if we found that two phenomena, not in immediate causal relation, such as pressure and attraction were occurring at the same place and at the same time; just as we are necessitated to postulate two psychical continuants when two psychical events, not in immediate causal relation, occur within the same period of time.

§ 7. In transeunt causality, as so far expounded, we conceive two continuants—which in the first instance are to be physical—in causal connection with one another; in such wise that the alterable 'state' of the one continuant is attributed as effect of its alterable relation with the other. This conception of transitive causality gives significance to the antithesis 'agent-patient.' That continuant whose 'state' is occasioned by its relation with the other continuant is termed (in this connection) patient, and that continuant whose relation to the former occasions the state is termed agent. Logicians who have rejected the antithesis between agent and patient have done so on the ground that every agent is at the same time patient and every patient is at the same time agent. But, even, if this were universally the case, the distinction would remain; since the state of the one continuant is effect of its relation with the other continuant while the concurrent state of the other continuant is effect of its relation with the

former. We can always distinguish between the one cause which occasions *its* effect and the other cause which occasions *its* effect. Hence, I should substitute for Kant's three categories of relation: Continuant and State; Cause and Effect; Agent and Patient.

Several points in the consideration of transeunt and immanent causality must be noted.

(a) Processes which are immanent to a whole system of interacting continuants may always be regarded as entailing transeunt causality between the parts of the whole system. This aspect of causality is familiar to the student of Physical Science. Or—to express the same principle in converse form—if we primarily conceive of interaction between parts of a system as exhibiting transeunt causality, we may (without contradiction) express our formulae in terms of causality immanent to the whole. Physics is at first provisionally monadistic, but it becomes increasingly monistic, in the sense that the entire range of physical phenomena come to be systematised as immanent to the whole. This reduction of the whole of physical reality to a self-contained system by no means precludes the exposition of details in terms of transeunt causality.

(b) Now, although a monistic form may be given to the system of all physical reality, psychical reality remains essentially pluralistic, and cannot be formulated monistically. In a certain sense, physical reality exhibits the kind of causality that is termed transeunt and no physical causality is strictly immanent. This is because the ultimate constituents of matter—if there are ultimate constituents—have, so to speak, no insides. A physical event must always and can only be described as a

changing or unchanging spatial relation of one thing to another,—the ultimate 'thing' having no inner 'states' which can be said to change or to remain unchanged. Hence, the immanency ascribed to the processes occurring within a mentally isolated material 'body,' is only immanency relative to processes occurring within other mentally isolated material 'bodies.' Nevertheless the conception of immanency cannot be eliminated in the formulation of physical laws; because the effects upon one body due to transeunt action from another are modifications of what *would be happening* within the body were no such transeunt causality in operation. Hence, the analysis of transeunt process always entails reference to immanent process; yet the converse (as it seems) does not universally hold; that is to say, it seems that purely immanent processes occur within the experiences of a single Experient (Psychical Continuant), though perhaps never within the happenings of a single Occupant (Physical Continuant).

§ 8. The more general problem in regard to transeunt and immanent causality relates to the modes in which the two forms operate in conjunction with one another. When any complete event is described in terms both of transeunt and of immanent causality, it would appear that, in transeunt causation, the cause-event and the effect-event are simultaneous; but that, in immanent causation, the cause-event always precedes the effect-event. This view is in direct contradiction to the prevailing view amongst philosophers who profess to attach scientific significance to the antithesis between the transeunt and the immanent. Illustrations in support of my contention will be found in the body of my work,

where the temporal relations between cause and effect are discussed. Where cause precedes effect, as in immanent causality, I hold, in agreement with other philosophers, that there is no temporal gap between the two; they are strictly contiguous or as Dr Broad expresses it *adjoined*. Similarly, in transeunt causality, so far as spatial-relations between the two concerned continuants can be assigned, strict spatial contiguity goes along with temporal co-incidence. The above account must be understood to be preliminary and in a sense provisional; for, on further investigation, it will be seen that the simple principle that I have laid down must be partially modified.

§ 9. The views advanced in Part III on the problem of mutual interaction between 'mind' and 'body' may here be sketched in outline; and it should be said at once that I adopt the common-sense dualistic position and am, therefore, largely concerned with reconciling this position with the claim of science to have succeeded in formulating psychical and physical processes in general but precise terms. The common-sense view expressed briefly is as follows. Certain physical processes occur in accordance with purely physical laws and are unaffected by 'mind'; and similarly certain psychical processes occur in accordance with purely psychical laws and are unaffected by 'body.' Again, there are critical instants when a physical cause occasions a psychical effect which I shall term a *sensation*; and there are critical instants when a psychical cause which I shall term a *volition* occasions a physical effect. Of these last two cases, the former I shall refer to under the heading physico-psychical causality; the latter, under

the heading psychico-physical causality. Since sensations (immediately occasioned by a physical cause) often engender psychical processes terminating in an act of volition which in its turn initiates a physical process; and since this latter sooner or later produces a physical consequent which, at a critical instant, occasions a sensation, the whole system of action and interaction assumes a cyclic form. In such cases, action initiated from either side is followed by reaction initiated from the other. But there is no reason to suppose that the cycle is in all cases completed. On the contrary, some stimuli which initiate modification of sensation are not followed by a consequent volition which initiates modification in the physical world; and some volitions which initiate modification in the physical world are not followed by a consequent stimulus which initiates modification of sensation. Action followed by reaction is probably the exception rather than the rule.

The cyclic processes may be roughly schematised as exhibiting, *alternately*, transeunt and immanent process. The Greek letters  $\phi$  and  $\psi$  indicate respectively 'physical' and 'psychical' occurrences, and an arrow stands for 'causing' as also for 'preceding.' Thus:

$$(1) \quad \phi_a \rightarrow \psi_1 \rightarrow \psi_2 \rightarrow \phi_b,$$

$$(2) \quad \psi_a \rightarrow \phi_1 \rightarrow \phi_2 \rightarrow \psi_b.$$

Here the action  $\phi_a \rightarrow \psi_1$  is followed by the reaction  $\psi_2 \rightarrow \phi_b$ , and the action  $\psi_a \rightarrow \phi_1$  is followed by the reaction  $\phi_2 \rightarrow \psi_b$ . While, moreover, these actions and reactions illustrate *transeunt* causality, the intermediate processes  $\psi_1 \rightarrow \psi_2$  and  $\phi_1 \rightarrow \phi_2$  I shall speak of as *immanent*.

In case (2), the relation of the originative volition  $\psi_a$  to the terminal sensation  $\psi_b$  illustrates 'purpose.'

In case (1), the relation of the physical occurrence  $\phi_a$  (which initiates the cycle) to the physical occurrence  $\phi_b$  (which terminates the cycle) raises a general problem which is as yet without any unanimously accepted solution. This problem must be approached from a new side.

The problem next immediately before us is that of psycho-physiological parallelism. The term 'parallelism' is the well-known figurative equivalent for one-one correspondence or one-one correlation. But, unfortunately, it is used with further implications of meaning, two of which are in flat contradiction with one another. In philosophical usage, parallelism is generally understood to *deny* causal relation between the psychical and physiological correspondents; but, in Science, no such denial is implied (except of course by those scientists who reject causality altogether and substitute invariability). Now the grounds for maintaining parallelism in the philosophical sense have nothing whatever in common with those for maintaining parallelism in the scientific sense. In fact, at least as regards neural and sensational processes, most uninstructed persons accept scientific parallelism and would (if it occurred to them) deny philosophical parallelism. They would say that, inasmuch as variations in sensation correspond to variations in neurosis (as they are informed by competent scientists) the former variations are certainly caused by the latter.

§ 10. Here it is to be noted that the scientific assertion of correspondence is one-sided, whenever (as seems inevitable) the notion of causality is superimposed upon that of invariability. Impartial correspondence would assert that, just as the causal antecedents of a sense-

stimulus—which occasions a modification of sense-experience—are purely physical, so the causal antecedents of a volition—which occasions a modification in the physical world—are purely psychical. Scientists, however, mostly appear to maintain that it is a mere illusion to suppose that the processes of desire or feeling and cognition or thought which terminate in a volition are causally operative. They maintain that the really operative causality resides in the neural process which, in accordance with the correspondence theory, accompanies the conative and cognitive experiences. In short, whenever the psychical processes  $\psi_1, \psi_2, \psi_3, \dots$  follow one another in a temporal and invariable order, this is so *because* the physical processes  $\phi_1, \phi_2, \phi_3, \dots$  follow one another in a temporal and invariable order. They, thus, tacitly maintain a one-sided operation of transeunt causality. They assert that the sequence  $\phi_1 \rightarrow \phi_2 \rightarrow \phi_3$  constitutes the cause of the sequence  $\psi_1 \rightarrow \psi_2 \rightarrow \psi_3$ , and this assertion entails that the sequence  $\psi_1 \rightarrow \psi_2 \rightarrow \psi_3$  never constitutes the cause of the sequence  $\phi_1 \rightarrow \phi_2 \rightarrow \phi_3$ . Adapting our previous schematisation to the present problem, the scientists' view would be indicated thus:

$$\begin{array}{ccc} \psi_1 & \rightarrow & \psi_2 \\ \uparrow & & \uparrow \\ \phi_1 & \rightarrow & \phi_2 \end{array}$$

in contrast with

$$\begin{array}{ccc} \psi_1 & \rightarrow & \psi_2 \\ \downarrow & & \downarrow \\ \phi_1 & \rightarrow & \phi_2 \end{array}$$

where the vertical arrows (in both cases) stand for transeunt causality.



Of course, if causality were excluded altogether, so that the vertical arrows stood merely for simultaneity and the horizontal arrows merely for sequence, then there would be no relevant distinction between the two alternative modes of representing the facts. Now, the view of alternate action and reaction is partially expressed by saying that, in some cases  $\phi_1$  and  $\phi_2$  respectively cause  $\psi_1$  and  $\psi_2$ , while in other cases  $\psi_1$  and  $\psi_2$  respectively cause  $\phi_1$  and  $\phi_2$ . That is to say in cases where  $\psi_1, \psi_2$ , etc. stands for a sequence of sensations then these are related to the sequence of neural processes  $\phi_1, \phi_2$ , etc. as effect to cause. But in cases where  $\psi_1, \psi_2$ , etc. stand for a course of conative and cognitive deliberation, then (if this course is accompanied by any discoverable physiological processes *corresponding* to the course of the psychical processes)  $\psi_1, \psi_2$ , etc. are related to  $\phi_1, \phi_2$  as cause to effect.

In Part III a still bolder view is put forward: viz. that just as there are countless cases in which physical processes do not immediately occasion any psychical processes whatever, so there are cases in which psychical processes do not immediately occasion any physical process whatever. This view may be termed *impartial dualism*. Or—expressing the same view in metaphorical but familiar language—what is maintained is that man is a genuinely causal agent in reference to which his bodily organism serves directly and materials outside his organism indirectly as *instruments of his will*. On this view, a volition is immanently caused by such purely psychical processes as feeling, desire, knowledge and thought to which there are no neural or physiological correspondents.

§ 11. Before attempting to give direct evidence in support of the theory of impartial dualism, the scientific objections to this view must first be met. Physical Science claims that, in such a cycle as  $\phi_a \rightarrow \psi_1 \rightarrow \psi_2 \dots \rightarrow \psi_n \rightarrow \phi_b$  theoretically completed knowledge would be able, from the physical nature of  $\phi_a$ , to infer the physical nature of  $\phi_b$ , apart from any reference to the intermediate psychical occurrences  $\psi_1, \psi_2, \dots \psi_n$ . The chain of events would assume the form  $\phi_a \rightarrow \phi_1 \rightarrow \phi_2 \rightarrow \dots \rightarrow \phi_n \rightarrow \phi_b$ , where  $\phi_1, \phi_2, \dots \phi_n$  would represent assignable physiological processes occurring within a given bodily organism. Now the impartial dualist may fully admit this contention of the physical scientist and yet adhere to the view which attributes genuine causality to the 'mind.' For, the initial cause  $\phi_a$ , which operates from without the particular organism, does not enable science to infer the terminal effect  $\phi_b$ , without consideration of the *special* sequence  $\phi_1, \phi_2, \dots \phi_n$  which varies according to the *special* nature of the organism. The form of response or reaction set up in one organism (expressed by  $\phi_1, \phi_2, \dots \phi_n$ ) differs from that set up in another. These differences must be taken into consideration if the specific nature of the effect  $\phi_b$  is to be inferred. We must causally account for the differences in the intra-organic processes as between one organism and another. This account will entail reference to the *past* history of the individual organism and of its ancestors. But what is the nature of the cause that stamps upon this or that organism its own special mode of organic response? This speciality of response can be predicted, by means of ascertained rules of uniformity framed in purely physical terms; but *why* are such or such physical antecedents invariably followed by such

or such physical consequents? The character stamped upon each organism—by reference to which alone physical effects can be inferred from physical causes—may be the consequent of psychical processes, operating in such invariable modes as (theoretically at least) can be formulated in terms of physiological habits or trends or properties. The supremacy of physical law within the whole range of the physical is not hereby overthrown when mind is taken to be a genuinely efficient agent; for the notion of law may imply mere invariability, whereas that of an agent implies causality.

§ 12. A consideration of the different ways in which invariability and causality may be logically related gives rise to some questions of the greatest philosophical importance. In some cases, we have well-assured ground for asserting invariability, and from such assurance venture precariously to infer causality. In other cases, we have well-assured ground for asserting causality, and from such assurance venture precariously to infer invariability. The former type of case is that in which our main reliance is upon the accumulation of wide and varied instantial evidence; the latter, that in which our main reliance is upon the precision and accuracy with which we can analyse single instances. The distinction between these two types of logical procedure is, I believe, roughly illustrated in many regions of scientific enquiry. But I wish to maintain that this logical distinction can be applied as a ground of division between two departments of knowledge. By direct introspection, I feel assured that I can assign the cause of any one of my acts of will; but it is only with considerable doubt that I should venture to formulate rules in accordance

with which I invariably act. In virtue of this assurance I maintain that, in willing, I am both free and determined: *determined*, because my volition is not uncaused; *free*, because the immediate causal determinants of my volition are within my own consciousness.

Causal determination of the will cannot be based on the ground of any observable uniformity of behaviour on the part of myself or of mankind in general or of animals. This is partly because no universally applicable rules of behaviour can be formulated; but, more obviously, because I do not know in what precise points the determining antecedents of one action agree with or differ from those of another. In order to formulate rules of behaviour or conduct, I must obtain accumulative evidence upon which a precarious generalisation may be inductively grounded; and, when all that is conceivably possible has been carried out by inductive procedure, my reliance rests ultimately upon the direct assurance of causal determinism yielded by introspection.

§ 13. The above analysis is open to the charge of extreme *naïveté*. But, before attacking my position on this or other grounds, I ask my readers to note that my account of the will differs in some important respects from those given by others. Many disputants on the subject of freedom of the will have put determinism and freedom in antithesis, whereas the true antithesis is between determinism and indeterminism. This latter antithesis was (I believe) first explicitly put forward by Dr G. Ward, who was still more explicitly followed by Pearse and W. James. Sidgwick declares that in immediate consciousness we are assured of freedom, but he goes on to maintain that the determinism that

seems to be almost demonstrated by a sort of induction contradicts the freedom that is introspectively revealed. Again, many writers who reject determinism, interpret determinism as being materialistic:—a view which I absolutely disclaim. Again Mill and others reject freedom on the ground that it assumes the *effects* of volition to be known *a priori* without recourse to experience; whereas the freedom which I maintain entails rather direct knowledge of the immediate *causes* of volition. The knowledge of which I have direct assurance is a knowledge of the purely psychical phases such as desire and cognition of which I can become aware by retrospective or introspective attention; and these factors present themselves to me as cause of this or that volition. I am quite ignorant of the physiological processes which issue in an overt physical movement; and it is only after actual experience that I can foresee the more or less remote physical effects of any act of will, as is abundantly established by psychological enquiry. And again it is only by means of an extended experience that I can venture to generalise with respect to the volitions which will follow upon any recurrence of the same externally presented conditions, since the intensity of my desires and the determinateness of my cognitions are subject to alterations in the course of time.

One other frequent misrepresentation of the question under dispute must be mentioned. It is alleged against the determinist that he has falsely attributed to the will a kind of causality which is borrowed from the mechanical type of causation appropriate only to physical phenomena; whereas, in truth, as history proves, it is the type of causation exhibited in human volitions that

has been borrowed and falsely applied to physical phenomena.

§ 14. Some justification is needed for my devoting so large a space to the detailed discussion of such psychological or metaphysical topics as freedom and determinism in a work professedly logical. My excuse is that the psychological, metaphysical and logical aspects of these problems have not been properly disentangled; and that it is only by bringing these aspects into close connection with one another that we shall succeed in getting to the root of the matter. Many empirical psychologists have explicitly put forward the view that, whether or not freedom, in some metaphysical sense, is to be attributed to the will, at any rate psychologists must work on the hypothesis of determinism. In this way, they preclude any discussion as to whether psychological determinism is or is not incompatible with metaphysical freedom. Or again: Kantians have tried to reconcile transcendental freedom with empirical determinism. But this attempt needs a preliminary discussion of the logical relation between freedom and determinism; and, moreover, attributes freedom to the transcendental ego and determinism to the empirical ego. Now, in a philosophical treatment of such scientific conceptions as those of substance and causality, there is no place for a transcendental ego or any species of *Ding an sich*. The freedom attributed by science to the will is empirical in just the same sense as that in which determinism is attributed. What causally determines any act of volition is a temporal event or process manifesting the character of the psychical agent, just as what causally determines a physical consequent is a

temporal event or process manifesting the character of physical agents.

In order, then, to present a consistent and comprehensive view of the philosophical principles underlying scientific constructions and inferences, it is necessary to examine in what way such conceptions as cause and substance and such antitheses as transeunt and immanent causality are actually employed in science. The form in which these conceptions enter into psychical science fundamentally agrees with and also fundamentally differs from that in which they enter into physical science. Problems of parallelism and interaction could not be fruitfully discussed—even in a preliminary logical survey—without entering into controversial detail when attempting to apply the logical points at issue to the scientific analysis of psychical and physical facts.

## CHAPTER I

### FACT AND LAW

§ 1. ASSERTIONS about the universe of reality fall into two distinct classes which may be designated (1) assertions of fact and (2) assertions of law:—where the terms fact and law are restricted to the sense in which, taken together, they include experientially certifiable propositions and exclude formal propositions. Other terms approximately synonymous to 'fact' and 'law' are 'concrete' and 'abstract,' or again 'categorical' and 'hypothetical'; but these terms are used too loosely to bring out the antithesis which rests really upon the fundamental distinction and relation between substantive and adjective. Although according to our analysis every proposition is to be interpreted in terms of both substantive and adjective, we may assert provisionally that in the abstract proposition or assertion of law, the adjective is the more explicit or solely explicit factor, whereas, in the concrete proposition or assertion of fact, the substantive is the more explicit factor. Assertions of fact may be statements either of a single fact, i.e. about a single substantive, or of several single facts summarised in a proposition which shall have the same factual nature as the several propositions of which it is a summary. Or again, a concrete proposition may express not a conjunction but an alternation of single facts, and in this case it will be of the same nature as the assertions that constitute the several alternants,

though less determinate than any one of them. In discussing the nature of a factual proposition then, we need only consider the proposition which expresses a single fact, without conjunction or alternation. The subject term of such a proposition, which denotes a pure substantive without adjectival characterisation, is best symbolised as *S*, and '*S* is *p*' will stand for a single assertion of fact where *p* is the adjective characterising the substantive *S*.

§ 2. The first difficulty about the proposition '*S* is *p*' relates to what we may call the referential problem: in other words, to what subject is the predicate *p* to be referred when we assert '*S* is *p*'? For, if the symbol *S* is non-significant—and, in default of any adjectival characterisation, it is difficult to see what significance it can have—then the proposition '*S* is *p*' cannot be intelligently distinguished from, say, the proposition '*T* is *p*' where *T* is equally non-significant with *S*. If we agree that '*S* is *p*' and '*T* is *p*' are *different* propositions, we may yet look beyond them for a common class to which both terms *S* and *T* belong. This common class is denoted by the wide term substantive used in its very general sense; hence, as a further interpretation of our formulae, the two propositions to be distinguished may be rendered in the forms '*This* substantive is *p*' and '*That* substantive is *p*.' The introduction of the terms 'this' and 'that' serves to show that substantives can be distinguished apart from, and independently of, any adjectival characterisation; so that, starting with 'this substantive' and 'that substantive' we may complete our predication by asserting of 'this' or of 'that' either the same or a different adjective. As I have stated elsewhere, I regard the

principle of distinction which is independent of characterisation as ultimately based on the psychological fact of separateness of presentment of the manifestations of reality. The predesignation 'a certain' best indicates this separateness of presentment; and thus the more adequate formulation of the factual proposition runs: 'A certain given manifestation is *p*.' The introductory indefinite being preparatory to the referential definite, we pass from the predesignation 'a certain' to the definite 'this' or 'that.' This transition is possible psychologically so far as we can identify and discriminate the *positions*, temporal or spatial, at which manifestations are presented in separateness; and such identification or discrimination of position is, I maintain, psychologically prior to any subsequent relating in space or time, no less than to all forms of qualitative characterisation. The significance of the word 'given' in our formula is two-fold; in the first place, it indicates all that is meant by the word 'real'; and in the second place, it anticipates the *general* nature of the characterisation which completes the predication. For what is given, otherwise called the determinandum, is presented under a certain determinable, symbolisable by the capital letter *P* corresponding to the little letter *p*. The process of thought being the further determination of the relatively indeterminate, a further amendment of the formula will be: 'A certain given *P* is *p*.' Those logicians who wish to introduce identity into their analysis of the proposition may be partially gratified by this recurrence of the same letter in both subject and predicate<sup>1</sup>; but the fact that, ultimately, the subject term

<sup>1</sup> See Part II, Chapter I, § 9.



represents *indeterminately* what is represented *determinately* in the predicate term, does not preclude the referential problem of the singular categorical proposition; a problem which has been met by the unique employment of the phrase 'a certain' which is preparatory to the definite 'this' or 'that.' So much for the factual proposition.

§ 3. Passing to the consideration of the abstract proposition or assertion of law, this may be expressed purely in terms of characterising adjectives, in the form '*p* determines *q*.' Here the word '*determine*' demands special consideration. In our account of the simple categorical statement of fact, we spoke of determination by thought, and to apply determination in this sense to our abstract proposition, we should have to combine the abstract assertion '*p* determines *q*' with the concrete assertion 'a certain *P* is *p*,' these two propositions together *determining* us to assert 'this *P* is *q*.' According to this interpretation of the word 'determine,' the abstract proposition may be said to express an *anticipatory* determination for thought; for it must be conjoined with the concrete proposition in order to determine any further assertion.

Many logicians have been satisfied with this merely epistemic account of the relation of determination—a position which is tantamount to identifying the thought relation implication, with the causal relation in its widest sense, as indicated by Hume's phrase 'objective nexus.' Here we may note that Kant, deliberately opposing Hume, took the relation of implication to apply only to thought in general, and to be the typical form of judgment corresponding to the category of causality, the

causal relation having validity in an objective sense. In this contention Kant undoubtedly aimed at distinguishing the subjective or epistemic from the objective or constitutive relation; but on this matter of the very first importance his view has been very variously interpreted. Of all the interpretations I shall adopt that in which the two conceptions of determination are most widely opposed. Before entering into the detailed analysis of this position, we must refer back to the epistemic distinction between experiential and formal certification. For example, an arithmetical formula, expressing relations between numerical adjectives, is one that can be formally certified apart from particular experiences. In contrast to this, any proposition which formulates a law of nature can only be certified ultimately by means of particular experiences. Now in Mill's use of the phrase 'empirical uniformity' there seems to me to be involved a fundamental confusion between the epistemic and the constitutive points of view which it is immediately necessary to remove. Epistemically understood Mill's phrase points to the ultimate data, namely observed instances, upon which the generalisation under consideration is based; and since he holds that all generalisations about natural phenomena are established on this same basis, there should be no distinction for him between empirical uniformities and causal laws. Mill nevertheless hints at an ontological distinction between these two kinds of uniformity where, for instance, he asserts that the method of agreement cannot prove causal laws; for if, as seems probable, in using this phrase he meant the emphasis to fall on the words 'causal law,' he must have had an

ontological distinction in mind; it is only if the emphasis were upon the word 'prove' that a purely epistemic point arises. The same confusion is apparent in his view that the causal relation involves not only invariability but unconditionality. In my own view this qualification of Mill's represents the ontological distinction between a universal of fact and a universal of law. Thus taking the two determinate adjectives  $p$  and  $q$  under the respective determinables  $P$  and  $Q$ , the factual universal may be expressed in the form 'Every substantive  $PQ$  in the universe of reality is  $q$  if  $p$ '; while the assertion of law assumes the form 'Any substantive  $PQ$  in the universe of reality would be  $q$  if it were  $p$ .' These formulae represent fairly, I think, the distinction which Mill had in mind; for my first formula may be said to express a mere invariability in the association of  $q$  with  $p$ , while the second expresses the unconditional connection between  $q$  and  $p$ . Or, as I have said in p. 252, Chapter XIV, Part I, the universal of fact covers only the actual, whereas the universal of law extends beyond the actual into the range of the possible.

§ 4. Now the introduction of the word 'possible' here requires us to summarise briefly the main senses in which the word is used in common thought and in philosophy:

(a) The possible may be understood as equivalent to what is capable of being construed in thought; in this sense it is equivalent to the conceivable. Now the effort to construe in thought an entity which has been expressed in verbally intelligible form can be analysed into a step by step process such that the combination of characters and relations constructed up to a certain

point may present to us some further character which our thought is *compelled* to assign to the construction. What then constitutes the impossibility of the proposed construction is the attempt to replace this further character, which we were compelled to predicate, by another character which is positively opponent to the former. It is this positive opponency between characters, therefore, which constitutes the genuine inconceivability upon which non-existence is to be maintained. In other words, the *impossibility* of some one mental construction is derivative from the *necessity* of a contrary or opponent mental construction. Let us take the most familiar example: the non-existence of a collection defined at the same time as *two plus three* and as *seven*, does not depend directly upon the impossibility of mentally conjoining these two numerical predications, but indirectly upon the necessity of conjoining the predication *two plus three* with the predication *five*, of which *seven* is a positive opponent or contrary. It is not a question of difficulty—amounting to an apparent impossibility—of making a thought construction in accordance with a verbal formula that constitutes inconceivability and gives the true test of non-reality; but rather the positive necessity of making some determinate construction opponent to the proposed construction.

(b) A second meaning of the word possible is quite easy to define; it relates merely to the limitations of knowledge: so that we say it is possible that such or such may be the case, meaning to express the quite simple fact that we are not, at the time, able to make a positive assertion concerning the truth or falsity of the proposed proposition. In this sense of the word

possible, there is nothing in the nature of the proposition itself, apart from person and circumstance, which can determine its being possibly true or not, and for it I prefer to substitute the word problematic. A special case of this type of possibility arises when an individual has in his possession knowledge of various truths which he has not combined in thought, so as to elicit by mere thought process some further truth. In default of this thought process, the proposition expressing this further truth is not known, and is therefore possibly true and possibly false for him. All the complicated formulae of mathematics and logic come within this class for the ordinary man who has not taken occasion, or who is intellectually incapable, of developing such knowledge. This consideration leads to a third meaning of possibility.

(c) Propositions may be said to be possibly true or possibly false, in an explicitly referential sense; that is to say, possibility here is a feature not intrinsic to the proposition itself, but only when considered in reference to some other body of propositions taken to be true. Any proposition, then, whose falsity or truth cannot be formally deduced from a given body of propositions, may be said to be possibly true and possibly false referentially to this body.

(d) The further meanings of the word possible are connected with the notion of natural law and its antithesis to what we have called fact. The general form of a law, exhibiting the constitution of nature, has been expressed 'If any substantive were characterised as  $p$  it would be characterised as  $q$ .' This proposition expresses a relation between the characters  $p$  and  $q$  indi-

cative of the nature of the world of reality. If any two characters  $x$  and  $y$  are *not* so related, then the conjunction of  $x$  with any opponent of  $y$  would be said to be a *possible* conjunction. When speaking of any fact or event as distinguished merely by spatio-temporal position from other facts or events, such terms as necessary or contingent cannot be applied. On the other hand, when we describe the event by an enumeration of certain adjectives or characteristics finite in number, and therefore non-exhaustive, the nomic distinction between the necessary and the contingent has significance relatively to such *description* of the fact, though not relatively to the fact. Thus the fact may be described as a  $pqr$  which is  $x$ . And so described it will be nomically necessary provided that any substantive characterised by  $pqr$  would be characterised by  $x$ ; but it would be nomically contingent if anything characterised by  $pqr$  were not necessarily  $x$ . Now the nomic necessity—anything characterised by  $pqr$  would be characterised by  $x$ —implies the factual universal that 'everything that is  $pqr$  is actually  $x$ '; whereas the nomic contingency 'anything that is  $pqr$  is not necessarily  $x$ ' does not imply the factual particular that 'some things that are  $pqr$  are not  $x$ '; i.e. the affirmation of law, or nomic necessity, implies the factual universal; but the negation of law—i.e. the affirmation of nomic contingency—does not imply the factual particular. The logicians who reject the contrast that I maintain between law and fact, identify in effect nomic necessity with the universal of fact, and nomic contingency with the particular of fact. The conflict between these two views is apparent in the special case in which a factual universal expresses only

a contingency; that is to say, when 'Every  $pqr$  is  $x$ ' goes along with 'Any  $pqr$  might be not- $x$ ': e.g. the merely factual universal that 'Every day is followed by night' is compatible with the statement of contingency that 'Any day might be not followed by night.' Now the possibility of joining these two statements depends upon *day* being defined by a definitely limited conjunction of characters; for, if our definition exhausted all the characters, it would render the sequence of night inevitable, and we should be confronted with a universal of law. Expressing this symbolically:—An event described merely as a  $pqr$  that is  $x$  may represent a contingency; though such an event could theoretically always be more fully described as a  $pgruvw$  which is necessarily  $x$ . It may appear, since by an adequate description a contingency thus becomes a necessity, that the notion of nomic contingency has therefore no application. But, if we consider precisely why the conditions  $uvw$ , say, have to be added to the conditions  $pqr$ , in order that  $x$  may necessarily follow, it is because  $pqr$  does not nomically necessitate  $uvw$ , and therefore that the relation of  $pqr$  to  $uvw$  is nomically contingent. Thus the abstract question whether the character  $x$  of the given event is necessary or not is unanswerable, since it is seen to be contingent relatively to the incomplete description  $pqr$ ; and necessary relatively to the complete description  $pgruvw$ . The philosophical justification of the principle under consideration requires the postulate that any character such as  $x$  manifested in a particular event is ontologically dependent upon an assignable—and therefore finitely enumerable—set of characters  $pgruvw$ .

§ 5. In further explication of the formal distinction between the assertion of law and the ordinary or factual universal, we require to define the class expressed by the phrase 'Anything that might be  $p$ '; for there are limitations to this class. Thus, if there is a law of nature that anything that may be  $x$  would necessarily be not- $p$ , then a thing defined as in the class  $x$  would be excluded from the class of things that might be  $p$ , and this class includes, not anything whatever, but only such things as have been defined by a character not necessarily precluding  $p$ . Now a class defined in this way is very different from an ordinary or factual class; for we cannot take a given case and say whether it belongs to the class or not; all we know of a thing whose character is determined as  $x$ , say (where  $x$  corresponds to 'Anything that might be  $p$ '), is that this character  $x$  would not, under the realm of natural law, prohibit its being  $p$ . The particular case in question might, however, have other characteristics which *would* prohibit its being  $p$ : thus, if it be characterised as  $x$  and  $y$ , where the character  $x$  does not prohibit  $p$ , while the character  $y$  does prohibit its being  $p$ , and we take the completed definition  $xy$  of the class to which the thing belongs, it could not possibly be  $p$ ; but if we take the incomplete definition, which includes only  $x$  and drops  $y$ , then we may assert of the thing that it might possibly be  $p$ . A concrete illustration will make this point clear:—Let  $x$  stand for a railway journey, and let us suppose further that in any actual railway journey the train travels with a brake. Now as far as the definition railway journey is concerned, there is nothing that prevents the train travelling without a

brake, and therefore any instance might be one of a brakeless train. If we now take *y* to stand for the precaution which actually prevents the train being brakeless, the complete determination of the case would preclude it from being one of a brakeless train. The proposition to which our illustration leads may be put in the form: 'Any railway journey with a brakeless train would be liable to accident,' and the force of this proposition obviously extends over what must be called a wider range than the whole class of actual railway journeys; if we assume as a matter of fact that all railway journeys use a brake. This illustration suggests a wide class of cases which indicate human foresight or prudence; and in all such cases the distinction between the nomically necessary and the factual universal is quite apparent. If any action that might be characterised as such or such would produce undesirable consequences, and if human conduct is actually determined by knowledge of such consequences, then, as a matter of fact, those actions will never take place. The nature of the actual occurrences is defined, on the one hand, by the circumstances which would make such or such conduct disastrous, and on the other hand by the knowledge on the part of mankind, of this fact. If the occurrence be defined only by the circumstances, we can say of it that it *might* be such or such; but if, to complete the determination of the case, we add the knowledge of the consequences on the part of mankind, then this complete determination prohibits the possibility of its being characterised as such or such.

An example resembling that of the brakeless train, is 'Any person caught trespassing on this field will be

prosecuted.' This proposition applies not only to the persons who have actually been caught trespassing and who have therefore been prosecuted, but to persons who have trespassed and have not been caught; for it is true of these latter persons, as much as of the former, that if they had been caught they would have been prosecuted. The application of the proposition, therefore, again extends to the possible, and is not restricted to the actually existent; though this illustration differs from the other inasmuch as there *are* cases of uncaught trespassers, whereas we supposed that no train travelled without a brake.

Another illustration of the same principle may be taken from the sphere of physics. Thus from such a formula as—'retardation varies as the coefficient of friction'—it follows that if the coefficient of friction were reduced to zero, the retardation would be zero. But in actual fact there is no instance in which the movement of one body over another does not entail friction, so that the above proposition applies over a range beyond actual fact. The point of importance, therefore, is that an assertion of this type may be scientifically established as true, while there may be no case presented in fact to which it is actually applicable. If propositions of this kind were interpreted as merely existential or factual, the actual non-existence of the class defined by the subject term would render it a matter of indifference whether one or any other predicate term were substituted. It follows that a merely factual or existential interpretation of this type of proposition is totally inadequate, and that to express its significance the proposition must be understood as applicable to the



wide range of what is possible, as contrasted with the narrower range of what is actual.

§ 6. There is a subtle case in which the notion of the epistemically possible and the contingent—i.e. nomically possible—are combined, indicated by the term potential in one of its applications. Thus using the symbols employed above, we may know of a particular object that it is *pqr*, and that being such it may also be *x*; and further that if it were *uvw* as well as *pqr* it would necessarily be *x*. In this sense we may say that its being *x* is hypothetically necessary—a term which Mr Bradley uses to define the possible. Here the force of the word *hypothetical* is purely epistemic, and as thus applied it means that we do not know whether the thing is or will be *uvw*, knowing only that it is *pqr*. The term *necessary*, however, is used ontologically or nomically, and means that anything that is *pqruvw* would necessarily be *x*. Now I have to maintain that, given *pqr*, *x* cannot be said to be hypothetically necessary unless it is possible that anything that is *pqr* may also be *uvw*. We cannot therefore define the possible as equivalent to the hypothetically necessary, because the proposed character *x* is not even hypothetically necessary unless the junction of *pqr* with *uvw* is itself nomically possible. The meaning of the term potential, then, when the given thing, known to be *pqr*, is said to be potentially *x*, involves first epistemic possibility, i.e. we must not know that it is *uvw*; and secondly nomic possibility, i.e. anything that is *pqr* may be *uvw*. The most important use of the term potential coming under this wide head requires reference to a continuant subjected to transeunt causality. Thus to

say of a solid strong body that it is potentially capable of resisting a certain measurable degree of pressure implies, from the epistemic point of view, that our known data do not include the knowledge that such a big pressure will be actually applied; that such a force may be applied is therefore hypothetical, or more precisely, epistemically possible. But further, to convey the full significance of potential resistance, this epistemic possibility must be combined with the negative fact that there is nothing in the laws of nature, and in particular in the character of the body itself, which would prevent this large force being applied. Shortly, then, the potential resistance of a body means epistemically that we do not know whether a certain force will be applied or not; and ontologically or nomically, that there is nothing in the nature of things to prohibit such force being applied.

## CHAPTER II

### THE CRITERIA OF PROBLEMATIC INDUCTION

§ 1. THAT induction is the inverse of deduction is a truism; but it is worth while to develop this truism in its various aspects. In its simplest form this inverse relation is exhibited by the change of place of premisses and conclusion, for roughly deductive inference consists in the passage from 'All *P*'s are *Q*' to 'Certain given *P*'s are *Q*'; and inductive inference in the passage from 'Certain given *P*'s are *Q*' to 'All *P*'s are *Q*.' Whereas this deductive inference is formally demonstrative, the inductive inference is obviously only problematic, and in general of a low degree of probability. The logical theory of induction may be developed by showing in what respects the degree of probability of such an inductive conclusion depends on the aggregate-nature of the instances examined.

The general procedure in an inductive process is as follows: certain given instances are noticed as being characterised by certain adjectives—say *P* and *Q*—and we proceed to look for other instances characterised by *P*, in order to discover whether they are also characterised by *Q*. In the preliminary stages of induction, where *P* and *Q* jointly characterise certain observed instances, the sole factor which decides us to search for other instances of *P* in order to discover whether they are *Q*, rather than for other instances of *Q* to discover whether they are *P*, is that we have observed instances

of *Q* which are not *P*, while so far every observed instance of *P* has been *Q*. In other words, we already know that 'Not all *Q*'s are *P*,' and therefore our enquiry is restricted to the question whether 'All *P*'s are *Q*.' The search for new instances to which we are thus prompted constitutes the preliminary process called *discovery*, and these instances are presented to us in one or other of two ways. They may either occur in the course of nature, and be discovered by active search in appropriate places and at appropriate times; or, on the other hand, we may have the means of producing them at places and in times where the course of nature, if uninterfered with, would not have exhibited such instances. These two kinds of active search are briefly denominated non-experimental and experimental: both imply activity prompted and guided by a definitive purpose.

§ 2. The use of experiment in discovery can only be accounted for by anticipating a discussion of what is to be understood by the somewhat vague term 'uniformity.' We speak of experiment as an interference with the course of nature; but we do not in any sense conceive that by such interference the laws or uniformities of nature are violated; for of the laws or uniformities of nature we may provisionally say that they do not prescribe the dates and places at which phenomena will occur, except so far as the dates and places at which other phenomena have occurred have been determined. If then the human will has the power of directly producing phenomena, the *course* of nature is modified, though the *uniformities* of nature are not infringed. The phrase 'uniformity of Nature' thus involves a

certain ambiguity: it may mean uniformity in the *course* of nature independently of man's interference; as, for example, in the continued elliptic motion of the planets, or in the upward convection of heat through the atmosphere; or it may mean the aggregate of the laws or uniformities which are obeyed even when there has been human interference; as, for example, in the construction and working of Foucault's pendulum, or in carrying a hot body from one place to another. Whether, as in the first case, there is no interference, or as in the second case, human purpose intervenes, the laws of gravity and of convection of heat are equally unviolated; and in either case, the phenomena observed will afford means for studying the uniformities in accordance with which the operations of nature take place.

Further, from a certain point of view, uniformity in nature holds even when man interferes; for nature includes man, and we shall here assume that voluntary action obeys laws which as such are psychological, and exhibit the nature of man himself. Thus, if we suppose the occurrence of a definite purpose to be the immediate cause of the time, place, and manner of a certain interference in the course of nature, the formation of this purpose may be assumed to have depended upon antecedent psychological conditions, and thus to exhibit the kind of uniformity which is characteristic of man in his capacity of voluntary agent. The use of man's power to interfere with the course of physical nature is prompted not only by the purpose of acquiring further knowledge, but also by utilitarian ends. Thus the face of the physical world is, at the present time, totally different from what it would have been if the laws of physical nature alone

had been in operation. We must therefore recognise a partial independence together with a mutual interaction between psychical and physical process, each following its own laws and also affecting the phenomena of the other. And this interdependence resolves the apparent paradox of scientific experiment, which consists in interfering with the *course* of nature, with the purpose of discovering more determinately the *laws* of nature.

§ 3. Returning now to the search for new instances; it may be assumed, in the simplest case, that all the instances of  $p$  so far examined have been discovered to be  $q$ ; or, rather, more precisely, that none of them have been discovered to be other than  $q$ ; for the character  $q$  may, in certain instances in which we have detected  $p$ , be beyond our power of observation. We thus arrive at an enumerative universal, 'All examined  $p$ 's are  $q$ ,' and this proposition constitutes the inductive premiss from which we venture to infer with a lower or higher degree of probability that 'All  $p$ 's are  $q$ .' Now this summary or enumerative premiss may have very different degrees of value as evidence for the universal; we will therefore proceed to sketch in outline the different tests by which its value may be estimated.

In the first place, if this positive premiss stood alone it would in general have very little value; only when it is combined with one or more complementary propositions which taken together mutually support one another, has it serious evidential value. To constitute such a complementary, a proposition must have as its subject term a substantive with characters opposed to that of the subject of the positive premiss. This intro-

duces the general notion of the determinable and its opposed determinates, and is explained in detail in the chapter on eduction, where I discuss the employment of intensional and extensional intermediaries. Here we will simply point out that, in calling a set of premisses complementary, we are extending or modifying the use of that term from the sense in which it denotes pairs of propositions like 'Every  $p$  is  $q$ ' and 'Every non- $p$  is non- $q$ ,' so as to cover an indefinite number of propositions corresponding to the indefinite number of values of  $P$  which have been correlated with an equal number of different values of  $Q$ . Thus the inductive premiss may be represented as a set of complementary enumeratives:

Every examined  $p$  is  $q$   
 Every examined  $p'$  is  $q'$   
 Every examined  $p''$  is  $q''$   
 .....

and corresponding to these premisses, the inductive conclusion may be represented as a set of complementary universals:

Every  $p$  is  $q$   
 Every  $p'$  is  $q'$   
 Every  $p''$  is  $q''$   
 .....

It must not be supposed that each several of these premisses constitutes by itself the evidence for the corresponding universal; on the contrary, the several premisses taken jointly constitute the experiential data upon which the strength of evidence for each of the several universals depends. This kind of compound induction then, which aims at discovering evidence that

the value of  $Q$  depends upon the value of  $P$ , does so by finding that in all examined cases the value of  $Q$  has varied along with variations in the value of  $P$ , and that it has been found constant whenever  $P$  was constant. Thus the notion of dependence has two sides: (1) that the constancy of the one variable entails the constancy of the other; and (2) that the variation of the one variable entails a variation of the other. The reader will note that the collection of data in compound induction of this type roughly resembles Bacon's Table of Degrees and, somewhat less closely, Mill's Joint Method of Agreement and Difference.

§ 4. In rough or pre-scientific induction of the kind just described, it is not assumed that we are dealing with simplex variables, nor even with complex variables that have been analysed into their simplex factors: thus on further analysis we might afterwards discover that (say)  $p = ab'c$ , that  $p' = a'b'c$ , and that  $p'' = ab''c'$ ; and similarly with  $q$ . This leads to the consideration of another criterion which affects the cogency of inductive inferences, viz., what will be called the criterion of specification. For example: in the inference from 'Every examined  $p$  is  $q$ ' to 'Every  $p$  is  $q$ ,' there is a liability to generalise too widely—a danger which is great in proportion to the indeterminateness of the subject character  $p$ ; hence the more specifically  $p$  can be defined, the less hazardous will be our generalisation. The question arises: *How* specifically the determinate character  $p$  must be defined in order to limit this generalisation? Now the different cases which we have examined will all have agreed in *certain* characters, while, as regards *other* characters, some instances will have differed from others. We may

therefore conceive of a certain conjunction of characters—say *abcd*—which characterise every examined instance, and by including all these characters in our definition of the subject term, we limit our generalisation within strictly logical bounds. In the case before us the conclusion will then assume the form 'Every *abcd* is *q*'; and this strict specification prevents us from inferring any wider generalisation such as 'Every *abc* is *q*,' or 'Every *bcd* is *q*,' or the still wider generalisation 'Every *bc* is *q*.' An elementary illustration will help to explain the force of this principle of specification. Common experience had afforded mankind in early times invariable evidence of unsupported bodies falling to the earth; if from this they had inferred that all unsupported bodies would fall to the earth, they would have neglected a character common to all the observed instances—namely that of proximity to the earth; their generalisation ought therefore to have been restricted to the statement that every unsupported body *in proximity to the earth* would fall. There are many phenomena which can be observed by man in a region of space limited in some such way as this; and hence the generalisations based upon such observations should be limited to the regions in which the character is manifested. Of course this does not mean that natural phenomena are dependent upon absolute spatial conditions, but only that there may be material bodies, occupying particular regions of space, upon which the phenomena depend. The same applies to periods of time: absolute dating in time does not affect natural phenomena; but there may be types of events occurring within certain periods of time, upon which other occurrences within those

periods depend. Thus, under the general principle of evolution, the forms in which the uniformity of nature is manifested will be very different at different periods; it would therefore be invalid to infer, from the recorded evidence that nations throughout history have been either preparing for or actually engaged in war, that this will be the case in future: not because absolute time, any more than absolute space, is relevant for the uniformities of nature, but because the occurrences within a particular period of time causally affect other occurrences within that period; just as material bodies within a certain region of space causally affect other material bodies within that region.

The principle of specification can only be approximately realised in practice; for practically it demands that the instances examined shall agree with one another in no characters over and above those which are used to define the range of the generalisation. But if all the instances of *abcd* for example, agree in only one or two other characters, say *uv*, our generalisation, though it ought strictly to be limited to the narrower class *abcduv*, may perhaps be safely extended to the whole class *abcd*. The generalisation approximates to certainty in proportion as the additional characters common to the examined instances decrease in number. Thus the principle of specification, expressed in familiar language, demands that an assortment of instances designed to establish a generalisation, should be as varied as possible within the range defined by the characters comprised in the subject term. In this form it is seen to be practically equivalent to the principle underlying the method of agreement, which requires that the instances



upon which a generalisation is based should exhibit together the maximum of difference or of variety.

§ 5. The principle of specification applies to a single enumerative such as 'Every examined *abcd* is *q*'; but we have seen that the condition for the highest degree of probability is that the generalisation in question should be supported by a set of complementary enumeratives, and we proceed to consider what relations should subsist between these several enumeratives. The requisite condition in this case is that, under the several different enumeratives, the instances examined should agree with one another as closely as possible in all characters other than those in which they are known to differ. Thus within the *same* enumerative, the instances should *differ* from one another as far as possible in the other characters; and under *different* enumeratives, they should *agree* with one another as far as possible in the other characters. This latter requisite approximates in practice to Mill's method of difference, application of which demands that the instances examined should agree as far as possible. We thus have two complementary criteria, the one requiring *variety*, and the other *similarity*. These principles express the common practice of the uninstructed mind; we can only justify them when we enter into the relation of probability to induction.

§ 6. So far we have not referred to the number of instances examined as a criterion affecting the strength of evidence. In point of fact mere number does not directly strengthen the instancial evidence; its importance depends upon variety; and number counts only because, by increasing the number of instances under the method of

agreement, their variety is *probably* thereby increased. Corresponding to number, what is required for instances under the method of difference may be denominated proximity; not because mere proximity in time or space is important, but because instances which are either temporally or spatially close to one another will probably agree in many characters which it may be impossible for us to analyse. Thus the analogous criteria of number and proximity are both only inferior substitutes for analysis. When under agreement we cannot analyse sufficiently the characters of the instances to enable us to assert difference between them in many respects, we have to rely upon mere *number* of instances, which are presumed to secure a probable maximum of difference. Similarly under the method of difference, when the elaborate analysis required to enable us to assert agreement in many respects between the instances is impracticable, we have to rely upon mere *proximity* which is presumed to secure a probable maximum of agreement. The term proximity here is to be understood to include besides what would be literally called spatial or temporal proximity, also reference to the same agent whose conditions are varied from instance to instance.

§ 7. The fact that the criteria of number and proximity are mere inferior substitutes for the more direct criteria of variety and similarity, at once suggests that the evidential value of examined instances really depends upon the extent to which our analysis enables us to assert agreement or difference in the characters of the compared instances; and the probability of a generalisation therefore varies with the degree of precision with

which we are able to define the characters of the instances examined. This consideration throws light upon Mill's problem 'why, in some cases, a single instance is sufficient for a complete induction, while in others, myriads of concurring instances, without a single exception known or presumed, go such a very little way towards establishing a universal proposition.' Speaking in terms of mere number, intensional number is of much higher value than extensional number; that is to say the number of characters in which instances are known to agree and differ is of much greater evidential importance than the actual number of instances examined. But again the mere number of characters analysed is not directly important in itself, any more than the mere number of instances examined: the characters counted should be strictly *independent* of one another, and this requirement is exactly parallel to that which demands that instances examined should *vary* with one another. Any character whose presence *is dependent upon* the conjunction of a given set of characters adds nothing to their evidential value. And, similarly, any instance which *agrees with* a given set of instances in all the respects in which these agree with one another adds nothing to *their* evidential value. Therefore whenever, in order to construct the intensional criterion for problematic induction, we count characters which are not known to be independent, we are relying upon the likelihood that a good many of them *are* independent. And when, to constitute the extensional criterion, we merely count instances which are not known to be pertinently different, we are relying upon the likelihood that a good many of them *are* pertinently different.

§ 8. The above parallel applies to instances and their characters under what may be in general called the method of agreement, where the generalisation refers to the characters in which all the instances agree. A similar parallel may be drawn for the instances and their characters under what we may call the Joint Method or Method of Complementaries. Thus the larger the number of instances which agree in certain characters *relatively* to the total number of instances observed, the higher is the probability that some of these characters are dependent upon one another. And the larger the number of characters in which certain instances agree *relatively* to the total number of characters analysed, the higher is the probability that some of these instances will agree in other characters besides those analysed.

§ 9. We have said that the probability of a generalisation varies with the degree of precision or determinateness with which we are able to define the characters of the instances examined; this determinateness reaches its highest point when instruments of measurement can be employed; and this accounts for the high probability generally attributed to generalisations formulated in terms of mathematical conceptions. Thus a further criterion of probability rests upon the possibility of applying quantitative considerations. It may be pointed out that this criterion carries the one which precedes it one step further in the direction of determinateness: the earlier dealt with the number of characters belonging to different determinables; the later deals with the number of determinate characters distinguishable under the same determinable.

§ 10. The last criterion to be mentioned virtually sums up and further organises all the preceding criteria. It may be said to rest upon the comprehensive complexity with which a law correlating the examined instances can be formulated. The generalisation of course consists in extending such a formula to unexamined instances; and if we have been able to define with comprehensive exactitude the kind of complexity exhibited in the instances compared, then the probability with which the formula may be extended to unexamined instances is commonly held to approach very nearly to certitude.

§ 11. In discussing the general nature of the method of problematic induction, the relations of agreement and difference are those which have figured throughout as the two forms of analytic comparison. From this point of view there can be no method of direct induction which might not be denominated by Mill's phrase, the Joint Method of Agreement and Difference. It may be useful, therefore, to draw up a two-dimensional scheme corresponding to the two relations of agreement and difference, which will enable us to figure this method in imagination. Instead of speaking of instances under the method of agreement or difference, we shall speak of instances under the relation of relevant agreement or of relevant difference. Instances which relevantly agree will be figured in a set of parallel columns; and instances which relevantly differ, in a set of parallel rows. It will be assumed that of the instances schematised, those which relevantly agree with one another will have irrelevantly differed from one another as much as possible; and those which relevantly differ from one

another will have irrelevantly agreed with one another as much as possible.

| $p$                      | $p'$                      | $p''$                      |
|--------------------------|---------------------------|----------------------------|
| $(abcd)u v w \dots$      | $(abcd)'u v w \dots$      | $(abcd)''u v w \dots$      |
| $(abcd)u v' w' \dots$    | $(abcd)'u v' w' \dots$    | $(abcd)''u v' w' \dots$    |
| $(abcd)u' v'' w'' \dots$ | $(abcd)'u' v'' w'' \dots$ | $(abcd)''u' v'' w'' \dots$ |
| $(abcd)u'' v w' \dots$   | $(abcd)'u'' v w' \dots$   | $(abcd)''u'' v w' \dots$   |

In the above table any one of the columns contains instances which agree in the relevant characters  $ABCD$ , while they differ in some or other of the irrelevant characters  $UVW$ ; the column is supposed to be indefinitely extended so as to represent a collection of instances presenting the largest possible variety as regards the irrelevant characters  $UVW$ . Any one of the rows contains instances each of which differs from all the others in the relevant characters, while they agree as far as possible in all the irrelevant characters. It will be noted that the relevant characters have been bracketed, and the dashes affixed to the entire bracket—a mode of symbolisation which is intended to denote that the observer, being unable to analyse the complex  $ABCD$  into its simplex factors, may have been forced to regard the variations as pertaining to the complex as a whole. In comparing instances in the same row, therefore, the observer knows that there is *some* difference in the compared complexes, though he may not know to what special factor within the complex the difference attaches. If for the bracketed  $ABCD$ , a single letter—say  $M$ —were substituted, it would represent the observer's ignorance as to the nature of the factor he was varying—whether it was single, or if complex, which of the simplex factors, and how many

of them were being varied. Similarly of instances in the same column, the dashes affixed to the bracket *abcd* denote that the *same* variation is observed throughout all the instances, although the observer may not know to what simplex factor or factors it attaches. The different columns each representing a set of instances in the relation of agreement constitute what I have called a set of enumeratives; and to represent symbolically the generalisation inferred from this aggregate of instances, we must imagine the columns extended on the same pattern to infinity. The several columns constitute a set of complementary enumeratives, and are each problematically extended into a set of complementary universals; the final generalisation, representing each value of *P* as depending upon the correlated value of *ABCD*, including all these minor universals.

§ 12. A scheme of this kind does not of course represent the detailed criteria used to estimate the degree of probability of different generalisations; it suffices, however, as a basis for criticising certain popular views on induction. The word 'hypothesis' is often loosely used in this connection: all inductions, it is said, are hypothetical; or again every induction is based upon hypothesis. These two separate assertions are not—as is sometimes supposed—equivalent. That every induction is hypothetical presumably means that inductive generalisations must be accepted with some reserve as regards their probability; in short, that induction does not ensure certitude. Instead of speaking of induction as hypothetical, therefore, I prefer to speak of it as being problematic; meaning by this that inductive generalisations cannot be affirmed with certitude, but only with a

lower or higher degree of probability, depending upon the aggregate nature of the instances used to establish them. When on the other hand it is said that every induction is based upon hypothesis, 'hypothesis' means assumption; and the assumption referred to is some such proposition as that nature is uniform. Thus if it be held that the proposition that nature is uniform is not certainly true, but only probably true, then the degree of incertitude which attaches to the uniformity of nature must be attached to any induction whose validity depends upon the assumption of such uniformity. But again in this case I prefer to use the word problematic; for what is meant is that there attaches to the induction at least as low a degree of probability as has been attached to the proposition that nature is uniform.

A third meaning of the word 'hypothesis,' as it is used by Jevons, Sigwart and Bosanquet for example, when they assert that induction is based on hypothesis, requires separate discussion. In particular Jevons maintains that hypothesis is the first of the three stages in the completed inductive process, the second stage being called deduction, and the third verification. This use of the word hypothesis to denote the mere formulation of a generalisation which it is proposed to establish, is, in my opinion, totally unjustifiable. This so-called hypothesis or proposal constitutes the first stage in a process of which the third and final stage is called verification; in these two stages, therefore, reference is made to one and the same proposition, which is at first propounded as *to be* proved, and finally asserted *to have been* proved. In short the relation between the

two stages is precisely that which obtains in Euclid between the enunciation of a theorem which prefates a demonstration, and the termination of the proof which concludes with Q.E.D. But, if the word 'verification' is used here in its natural sense to imply that the conclusion of the inductive process is 'certified,' then it is meaningless to speak of induction as hypothetical in the sense of problematic. Jevons, however, maintains both these views: namely that completed induction ends with verification, and that induction involves an application of the theory of probability, thus rendering all generalisations problematic. If it is urged that I have taken the term verification too literally, and that all that is meant is that the generalisation is confirmed, and not actually verified, at this final stage, then the proper account of the process is simply that at the first stage a generalisation is accepted with a relatively low degree of probability, and at the last with a relatively high degree of probability; and there is no need to introduce the term 'hypothesis.'

My criticism of Jevons' account of induction extends, moreover, beyond his use of the notions of hypothesis and verification, to the stage of deduction which intervenes between them. According to his analysis, the proposition formulated in the first stage is taken as major premiss in a deductive process; the minor premiss being supplied by observation or experimentation. With these two premisses entertained in the mind as possible, a conclusion is drawn on purely deductive principles, referring in general to a single kind of case. Then, either with or without experimentation, we examine an instance of the type to which the deductive conclusion

refers: and if, on comparison, accordance is found between the conclusion deductively reached and the observation of this specially contrived instance, the final stage called verification is attained. In a certain sense, then, this deductive procedure includes all the purely *mental* part of what Jevons represents to be a complete inductive process. Thus the relation between deduction and induction in his scheme may be represented:

(*Deduction.*) If  $P_1$  and  $P_2$  are true, then  $C$  would be true;

(*Induction.*) If  $C$  is true, then  $P_1$  is true;

where  $P_1$  and  $P_2$  stand for the premisses and  $C$  for the conclusion. Now I wish to point out that the process of inference from  $P_1$  and  $P_2$  to  $C$  is in accordance with a demonstrative principle; but that inference from  $C$  to  $P_1$  cannot be governed by a demonstrative principle; it follows, therefore, that Jevons' attempt to reduce the principle of induction to the principle of deduction is vain. The explanation of this blunder is to be found, I think, in a confusion between hypothetical inference and categorical inference. The deductive process to which Jevons refers is a mere hypothetical inference, which might be written: the truth of the premisses *would* imply the truth of the conclusion; whereas the inductive process is a categorical inference, and might be written: the truth of the premiss *does* imply the truth of the conclusion.

§ 13. Let us now give credit to Jevons for the truth which lies concealed in his theory. It has been expressed by Whewell and others in the principle that the sole test of an inductive generalisation is accordance with facts. This principle I hold to convey a truth, but only a



partial truth; for in the first place it neglects the variable degree of probability to be attached to a generalisation based solely upon accordance with facts; and in the second place it neglects the variable degree of accordance which could be attributed to the relation of the generalisation to the facts. Combining these two neglected considerations, the principle of problematic induction may be restated in the following form:

*The degree of probability to be attached to a generalisation based upon facts varies directly with the degree of accordance between the generalisation and the facts.*

This maxim does not, of course, claim to be expressed with mathematical precision; the whole problem of the theory of induction is to define as precisely as possible what is meant by 'varying degrees of accordance.' Roughly, however, the main factors upon which such accordance depends are the number and variety of instances covered by the formula, and the determinateness with which the formula fits the facts.

1. With regard to the number of instances, the generalisation ranges over an infinite number of possible cases; hence the larger the number of observed facts found to conform with it, the higher its degree of accordance—on the score of mere number.

2. With regard to variety of instances, the generalisation claims to apply irrespective of circumstance; hence the wider the range of variety of circumstance in the instances observed, the higher will be the degree of accordance of the generalisation with the facts—on the score of variety.

3. With regard to determinateness, the degree of accordance is high in proportion as the generalisation

fits the facts closely and precisely. Thus, if a formula is comparatively indeterminate, then it cannot be said to accord closely with facts, even though it may cover a large range. For example, the generalisation that bodies falling to the ground move more and more rapidly as they descend may be confirmed by observing an actual increase of velocity, in which case a certain degree of accordance could be said to obtain between the formula and the facts. But if the formula asserts that for every second the rate of movement increases by approximately 32 feet per second, and by measuring the actual fall of bodies it is ascertained that the velocity of their descent does actually increase at this rate, the degree of accordance in this case may be said to be high—on the score of comparative determinateness.

## CHAPTER III

### DEPENDENCY AND INDEPENDENCY

§ 1. AGREEMENT and difference—the two principles upon which every method of direct induction ultimately depends—are notions which may be further expounded and more precisely defined by a logical analysis of the kind of proposition which directly expresses the data of observation. Such a proposition assumes the form: certain observed manifestations are characterised by the descriptive adjectives *mnpqr*, say. Now this form of proposition is—in two main respects—different from that with which we have been chiefly familiarised in logical teaching. In the first place, the familiar terms of quantity, such as ‘all’ or ‘some’ are omitted; and therefore one important aspect of induction is that it represents inference from a proposition concerning ‘certain cases’ to a conclusion about ‘all cases’. In the second place, the proposition expressing the data of observation does not distinguish between those characters that define the subject term and those that define the predicate term: that is to say, it does not assume the familiar form ‘Everything that is *p* is *q*.’ Hence in passing from the proposition that directly expresses the data of observation to the proposition that expresses the conclusion inductively inferred, two kinds of transformation occur. The first transformation is from ‘certain’ to ‘every,’ and depends upon the condition of *variance* amongst the manifestations recorded. The

second transformation is from a proposition containing no characterised subject term to a typical proposition which has a characterised subject term as well as a distinct characterising predicate term. This is equivalent to splitting up the conjunction of adjectives *mnpqr* into, say, *pqr* to constitute the characterising description of the subject term, and *mn* the characterising description of the predicate term—a separation which is rendered possible by distinguishing amongst the descriptive adjectives *mnpqr*, those which are independent of one another—viz. in our illustration *pqr*—from those which are inferred to be dependent upon the former—viz. *mn*. Only by combining these two transforming processes, therefore, can we infer from the inductive premiss ‘Certain manifestations are *mnpqr*,’ the inductive conclusion ‘All manifestations that are *pqr* are *mn*’; and the two essential conditions required are (1) for the transformation from ‘certain’ to ‘every,’ variance of the observed manifestations; and (2) for the separation of the subject characters from the predicate characters, establishment of independence amongst the several subject characters.

Closer enquiry into the first stage of this transformation shows that it takes place before any separation of the subject from the predicate characters; or rather, to express the distinction more suggestively, of the determining from the determined characters. Instances, collected on the ground of manifesting certain characters in common, will always have manifested other characters differing from instance to instance. These inconstant characters have been omitted or eliminated in our summary description *mnpqr* of the data of observation;

but it is in virtue of these omitted characters that the observed manifestations can be said to have the aggregate nature of variancy. Thus, in this preliminary process, we conceive the characters which maintain their cohesion as forming a constant combination which is incapable of being destroyed by variations in other concomitant characters, the cohesion being the stronger, the greater the degree of this variation. This process corresponds to the principle of agreement, which has two aspects: namely, the elimination of varying characters, and the retention of a combination of constant characters.

The result of this first transformation may be stated in the form: there is *some* relation of dependence amongst the characters  $mnpqr$ ; and the characters being regarded as a dependent conjunction, we are led to the second stage of our enquiry, viz. which of them are dependent upon the others? That is to say, we have next to discover amongst the characters in the constant conjunction, those which are *independent* of one another, and which therefore constitute the determining characters, by the conjunction of which the others are probably determined. Again we rely ultimately upon observation of instances, which, in order to lead to the separation of the characters  $pqr$  as independent of one another from the characters  $m$  and  $n$  as probably dependent upon them jointly, must have been of such a nature that wherever one alone of the characters  $pqr$  has varied, then  $m$  and  $n$  will have been found to vary; and wherever all the characters  $pqr$  were jointly constant,  $m$  and  $n$  were found to be constant.

§ 2. From this point onwards, our observations may

be conducted under so-called 'experimental conditions'; the object of enquiry being to discover the specific values of  $P, Q, R$  which determine specific values of  $M, N$ . The special value of experiment lies in the completeness and accuracy with which it enables the experimenter to define the relations of agreement and difference subsisting between the determining and determined characters in each of the several instances observed. To secure the highest degree of accuracy and completeness, it is generally necessary to have recourse to *experiment* in the strict sense in which it implies that we have been able ourselves to contrive the instances observed. In such a case the experimenter knows beforehand which characters can be taken, for his purpose, as *independent* of one another—in the sense that he can vary these at will, with the assurance that the others will remain constant—and which characters are to be taken as dependent upon the former, in the sense that he is awaiting their manifestation in ignorance as to whether they will prove constant or varying, or varying to this or that degree. Thus he determines with accuracy the determinate values  $p, q, r$  before the result of the experiment is known, and then measures with equal accuracy the determinate values  $m$  and  $n$ ; in this way ascertaining the precise effect which follows upon a precise cause, where previous to the experiment, both the cause and the effect were defined with comparative indeterminateness. But such exact experimentation presupposes the separation of the dependent from the independent factors; the dependent characters being those whose determinate values the experimenter wishes to learn as the *result* of the

experiment; while the independent characters are those the determinate values of which he knows *before* experimenting. As has often been pointed out, the experimenter may of course be mistaken on these points; and while varying one factor, be unintentionally varying another, which is causally dependent upon it. This mistake would involve the assumption that certain factors were independent which were in reality dependent; but the mistake which I propose next to examine is the supposition, or rather inference, that certain factors are dependent, when they are really independent. A fallacy of this kind arises only where it is impossible for the scientist to contrive a variation in the characters manifested in nature in constantly cohering groups.

§ 3. To illustrate such an incorrect supposition, let us suppose a variation in  $b$  in two instances symbolised as  $abcdq$ ,  $ab'cdq'$ . These concomitances might be analysed either in the form  $bcd \sim aq$  and  $b'cd \sim aq'$ , or in the form  $abcd \sim q$  and  $ab'cd \sim q'$ . In the first analysis  $a$  is taken to be dependent, and only  $bcd$  independent of one another; in the second analysis  $abcd$  are taken all as independent of one another. The fallacy that we are considering is the assumption that the former is correct when, in truth, the latter is correct; that is, the factor  $a$  has been falsely supposed to be dependent, when in reality it is independent. This incorrect analysis would lead us to infer, (1) since the variation of  $b$  alone entails no variation in  $a$ , that for all instances  $cd \sim a$ ; and (2) since the variation of  $b$  alone entails a variation of  $q$ , that for all instances  $bcd \sim q$ ,  $b'cd \sim q'$ . But these two inferences could not have been made from the correct analysis; that is, we could not

have inferred that  $q$  would follow from  $bcd$  for all values of  $A$ , or that  $q'$  would follow from  $b'cd$  for all values of  $A$ ; because, as far as the given instances alone are considered, the causal factors include the determinate value  $a$  along with the determinate values of  $bcd$ . Again the fact that  $A$  is independent of  $cd$  invalidates the inference  $cd \sim a$ , since, as a matter of fact,  $a$  might have had any value whatever concomitantly with  $cd$ . These inferences show that the employment of the figures of agreement and difference requires us to select beforehand those characters that can be properly regarded as independent of one another, as distinct from those which are dependent jointly upon them; for in both cases error occurs because the symbol  $a$  is placed on the side of the dependent factors, when it ought to have been placed among the independent. A more general form of exhibiting this same fallacy is to recognise independencies as being of certain numerical orders. Thus the false analysis  $bcd \sim aq$  represents the independency to be of order 3, when the correct analysis  $abcd \sim q$  shows it to be of order 4. Although it is always mistaken to assume an independency to be of a lower order than is actually the case, there is no objection to provisionally assuming it to be of a higher order than it actually is. In fact, the conclusions derived from the employment of the figure of agreement are those in which independencies provisionally assumed to be of order 4 say, are proved to be really of order 3, by showing that one of the supposed determining factors may vary without affecting the value of the determined factor, and may therefore be eliminated from the determining group. Another way of expressing the fallacy

under consideration, therefore, is to say that an independency is represented as of a lower numerical order than is correct.

It must be noted that the order of an independency is not absolute, but relative to the effect factor whose variation is under consideration. Thus if we suppose a further complication added to our original instance, this would assume the form  $a'bcd \sim pq$  which, by comparison with  $abcd \sim pq$ , leads, in accordance with the figure of agreement, to the elimination of  $a$  as inoperative upon  $p$ . This inference is correct on the assumption that as regards the effect  $p$ , the factors  $abcd$  constitute an independent group. In respect of the effect  $p$  alone therefore, the order of independence is 3, i.e.  $bcd \sim p$ ; while, with respect to the effect  $q$ , the order of independence was 4, since we could not eliminate  $a$  as inoperative in determining  $q$ .

## CHAPTER IV

### EDUCTION

§ 1. THE term eduction is chosen to describe the kind of inference which Mill speaks of as from particulars to particulars. In place of Mill's phrase, I should substitute inference from instances to instances, and in using the technical term eduction, I wish to point out where I agree with, and where I differ from, Mill's view of the relation of deduction to induction. If we consider the singular instancial proposition ' $s$  is  $p$ ,' it might stand first as a conclusion deduced from 'Every  $m$  is  $p$ ' and ' $s$  is  $m$ '; or secondly as a premiss which together with ' $s$  is  $m$ ' leads to the inductive inference 'Every  $m$  is  $p$ .' Now according to Mill, a more ultimate analysis of the deductive or syllogistic argument reveals it to be founded upon instances such as  $s_1, s_2, s_3, \dots$  which, being  $m$ , are also  $p$ , so that the universal 'Every  $m$  is  $p$ ' contributes nothing to the factual data upon which the syllogistic conclusion ' $s$  is  $p$ ' is based; and he leads the reader to assume that the single conclusion ' $s$  is  $p$ ' is established with the same force as the universal 'Every  $m$  is  $p$ ' from the instances of  $s_1, s_2, s_3, \dots$  that are  $p$ . At this point, however, I differ from Mill, and distinguish the type of inference which from 'certain  $s$ 's that are  $m$  are  $p$ ' concludes that 'Every  $m$  is  $p$ ,' from the type of inference which, from the same premisses, concludes that a further instance of  $s$  that is  $m$  is  $p$ . The former is called induction, and for the purpose of distinction, I



give to the latter the name of eduction—this term indicating that the conclusion merely goes outside the instances which constitute the premiss, while the inductive conclusion extends to all instances of an assigned character. Mill's statement that the evidence for the conclusion '*s* is *p*' is the same as the evidence for the universal 'Every *m* is *p*' is somewhat hasty; for on the surface it would appear that from the same evidential data the single conclusion can be drawn with higher credibility than the universal. For the present, however, we will postpone the question of probability, and define more precisely the nature of the instantial premisses upon which both the eductive and inductive procedure are based.

§ 2. In this connection we have first to criticise Mill's use of the term 'particular' in his analysis of the process of 'inference from particulars to particulars'; for he appears to assert that the inference to '*s* is *p*' is based merely upon premisses such as *s*<sub>1</sub> is *p*, *s*<sub>2</sub> is *p*, *s*<sub>3</sub> is *p*, where *s*<sub>1</sub>, *s*<sub>2</sub>, *s*<sub>3</sub> might stand for anything whatever; much as if we argued, because fire is red, and poppies are red, and Mr Webb's tie is red, that therefore the British Constitution is red. In short he has neglected the essential question of the mediating conception, through which we pass from given instances that are *p* to some new instance. This mediating conception is precisely equivalent to the middle term of the syllogism, and I therefore here represent it by the symbol *m*; only when it is known that *s* is *m*, and that *m* is a character common to *s*<sub>1</sub>, *s*<sub>2</sub>, *s*<sub>3</sub>, all of which are characterised by *p*, can we infer with any semblance of probability that the new instance *s* will also have the character *p*, which was

the proposed predicate in our conclusion. Thus, for inferring the proposed conclusion '*s* is *p*,' the minimum of instantial data required includes three propositions; viz., '*s*<sub>1</sub> is *m*,' '*s*<sub>1</sub> is *p*,' and '*s* is *m*,' where *m* and *p* are of the nature of adjectives or universals, while the instances themselves are of the nature of substantives or particulars. A simple example of such a triad of premisses is as follows:

Mars is a solar planet.  
The earth is a solar planet.  
The earth is inhabited.  
∴ Mars is inhabited.

Here the only known point of agreement between Mars and the earth is that they are both solar planets, and from this very slender relation of agreement we infer with the lowest degree of probability that Mars is inhabited, because we know the earth to be so. The probability of this conclusion is strengthened, the greater the number of characters in which Mars is found to agree with the earth; e.g. its being near the sun, and having atmosphere and vapour. It would be still further strengthened, if other solar planets besides the earth were known to be near the sun, to have atmosphere and vapour, and to be inhabited. The more complete process of eduction thus exemplified may be represented in the following scheme:

- (1) *s* is characterised-by *p*<sub>1</sub> and *p*<sub>2</sub> and ...*p*<sub>*m*</sub>,
  - (2) *p*<sub>1</sub> and *p*<sub>2</sub> and ...*p*<sub>*m*</sub> characterise *s*<sub>1</sub> and *s*<sub>2</sub> and ...*s*<sub>*n*</sub>,
  - (3) *s*<sub>1</sub> and *s*<sub>2</sub> and ...*s*<sub>*n*</sub> are characterised-by *p*,
- ∴ *s* is characterised-by *p*.

Thus, in eduction there are three summary premisses, containing (a) the summary term '*p*<sub>1</sub> and *p*<sub>2</sub>... and *p*<sub>*m*</sub>' which is adjectival; and (b) the summary term '*s*<sub>1</sub> and

$s_2 \dots$  and  $s_n$  which is substantival; besides the substantival term  $s$  and the adjectival term  $p$ , which occur in the conclusion. The mediating term ' $p_1$  and  $p_2 \dots$  and  $p_m$ ' will be denominated the *intensional* middle term, and the mediating term ' $s_1$  and  $s_2 \dots$  and  $s_n$ ' the *extensional* middle term,  $s$  and  $p$  being respectively the *minor* and *major* terms. The premiss containing  $s$  will be called the *minor* premiss; that containing  $p$ , the *major* premiss; and that containing neither  $s$  nor  $p$ , the *mediating* or *middle* premiss. The eductive scheme may be conveniently represented in the form of a chain, showing how the subject and predicate of the conclusion are linked up through the two mediating terms, thus:

|                                                  |                                     |
|--------------------------------------------------|-------------------------------------|
| Minor premiss                                    | Major premiss                       |
| $s \tilde{\chi} p_1, p_2 \dots p_m \tilde{\chi}$ | $s_1, s_2 \dots s_n \tilde{\chi} p$ |
| Middle premiss                                   |                                     |

where  $\tilde{\chi}$  = characterises, and  $\tilde{\chi}$  = is-characterised-by.

§ 3. Previous logicians have rather awkwardly contrasted inference by analogy with inference by induction—some regarding analogy as the basis of induction, and others taking induction to be the basis of analogy. In what sense these two terms are used is not clear, except that induction is understood to depend primarily upon the number of instances known to be characterised by a certain adjective; while the force of analogy depends upon the number of adjectives that are known to characterise a certain instance. But it is essential to insist that neither by accumulating instances alone, nor by accumulating adjectives alone, can any inference be drawn, and that inference of this type, by whatever name it may be called, is governed by principles which underlie both induction and analogy—requiring an intensional as well as an extensional link. For example,

no mere accumulation of instances  $s_1, s_2, s_3, \dots s_n$  that are  $p$  could give any probability that a new instance  $s$  will be  $p$ , unless  $s$  were known to have at least one character predicable of all these instances. And conversely, no accumulation of characters  $p_1, p_2, \dots p_m$  that are predicable of  $s$ , could give any probability that a new character  $p$  is predicable of  $s$ , unless  $p$  were known to be predicable of at least one instance having all these characters.

For the purposes of developing the principles of education in their relations to probability, a fundamental distinction must be made according to whether

(a) all the evidential data are in favour of  $s$  being  $p$ , or (b) some of the evidential data are in favour, and others unfavourable.

In case (a) the eductive process leads to an inductive inference whose conclusion is universal; in case (b) to an inductive inference whose conclusion is class-fractional. The remainder of this chapter will be limited to case (a)<sup>1</sup>.

Now the data favourable to the proposal  $s$  is  $p$ , fall into four heads:

- (1) Intermediaries  $p_1 \dots, s_1 \dots$ , such that  
 $s \tilde{\chi} p_1 \tilde{\chi} s_1 \tilde{\chi} p$
- (2) Intermediaries  $q_1 \dots, t_1 \dots$ , such that  
 $s \tilde{\chi} q_1 \tilde{\chi} t_1 \tilde{\chi} p$
- (3) Intermediaries  $x_1 \dots, u_1 \dots$ , such that  
 $s \tilde{\chi} x_1 \tilde{\chi} u_1 \tilde{\chi} p$
- (4) Intermediaries  $y_1 \dots, v_1 \dots$ , such that  
 $s \tilde{\chi} y_1 \tilde{\chi} v_1 \tilde{\chi} p$

These four cases will be recognised as all favouring the proposal  $s$  is  $p$ , because they constitute the different ways in which an even number of non-characterising links enter.

<sup>1</sup> Case (b) will be treated in the Appendix to Part III.

§4. If now all the evidential items are of one or other of these four kinds, then they all co-operate in strengthening the probability that  $s$  is  $p$ . Speaking generally, the larger the number of data of this kind that have been established, the higher is the probability that  $s$  is  $p$ . But certain conditions must be fulfilled in order that any apparently new evidence should actually strengthen the required probability. First as regards the extensional aspect of the evidence. The enumerated set of instances  $s_1 \dots s_n$  will count as  $n$  separate data, provided that every one of them such as  $s_n$  is uncharacterised by some of the adjectives that characterise all the remainder, i.e.

$$s_n \bar{\chi} q \bar{\chi} s_1 s_2 \dots s_{n-1}.$$

In other words, the instance  $s_n$  counts as one additional item of evidence provided that it increases the variety of the evidence; which it would fail to do unless it had, besides a certain nucleus of characters common to all the other instances, some character opposed to the common characters of the others. Secondly, as regards the intensional aspect of the evidence: The enumerated set of adjectives  $p_1, p_2 \dots p_m$  will count as  $m$  separate data, provided that every one of them such as  $p_m$ , does not characterise all the instances that are characterised by the remainder, i.e.

$$p_m \bar{\chi} t \bar{\chi} p_1 p_2 \dots p_{m-1}.$$

Hence an instance  $s_n$  adds to the weight of evidence when it constitutes a variation upon the other instances; and an adjective  $p_m$  adds to the weight of the evidence when it is independent of the other characters.

§5. We are thus enabled to establish certain principles regulating the strengthening force of evidential data.

In inferring from examined subjects that a given subject has a property characterising these, we rely upon the likeness of the new subject to those adduced, and the force of any such new instance varies with the degree of resemblance to the adduced subjects, and with the degree of unlikeness amongst the examined subjects themselves. The more remotely the latter differ from one another, the stronger is the evidence that they will agree with the new subject in further points, beyond those in which they are known to agree. In other words, the more varied or non-congruent with one another the accumulated subjects, the stronger the evidence in favour of a certain congruence. Similarly with regard to the intensional aspect of the evidence; the predicates  $p_1 \dots p_m$  which serve as intermediaries must be as independent of one another as possible, when used as evidence for establishing the dependence of a further proposed predicate upon these given predicates taken in conjunction. Summing up, then, for establishing a proposed congruence, the condition required is non-congruence amongst the examined subjects; and for establishing a proposed dependence, the condition required is non-dependence amongst the examined predicates.

These formulae represent the final inferred inductive conclusion in its two-fold universality; such universality being, on both sides, unlimited. In contrast to this unlimited universality, the evidence for such an inductive conclusion has to be exhibited in terms of what by observation and examination is *known*. Thus when all the subjects known to be characterised by  $p_1 \dots p_m$  are also characterised by  $p$ , we infer inductively that all subjects so characterised, will be characterised by  $p$ .

And when again all the predicates that are known to characterise  $s_1 \dots s_n$  also characterise  $s$ , we infer inductively that all the predicates which characterise this set will also characterise  $s$ . What we have here expressed in the form of a set of subjects generating a subject group, and a set of predicates generating a predicate group, is in effect equivalent to what was above explained as the intensional and extensional aspect of the eductive process.

§ 6. It will be observed that the instantial evidence for the proposition ' $s$  is  $p$ ' does not point to any one given  $s$  or  $p$ , but to any subject characterised by the conjunction of predicates  $p_1 \dots p_m$ , and any predicate characterising the conjunction of subjects  $s_1 \dots s_n$ . We can therefore eliminate the explicit symbols  $s$  and  $p$ , and consider only what we have called the intermediary premisses, summed up in the proposition  $s_1 \dots s_n$  are characterised by  $p_1 \dots p_m$ . This proposition will be called the summarised evidential datum, pointing to the conclusion that any unassigned subject characterised by  $p_1 \dots p_m$  will be characterised by any unassigned predicate that characterises  $s_1 \dots s_n$ . This summary evidence contains  $mn$  atomic data, each additional subject and each additional predicate in the two conjunctions counting as one.

We now proceed to consider the precise condition required in order that each of these subjects and each of these predicates shall count as one, in the estimate of the evidence before us. The condition that the predicate  $p_m$ , say, shall count as an additional item, is that there shall be some subject, say  $\sigma_n$ , such that

$$p_m \tilde{\chi} \sigma_n \tilde{\chi} p_1 p_2 \dots p_{m-1},$$

where there is also a subject  $\sigma_{n-1}$  such that

$$p_{m-1} \tilde{\chi} \sigma_{n-1} \tilde{\chi} p_1 p_2 \dots p_{m-2}$$

and so on. In other words, each additional predicate,  $p_m, p_{m-1}$ , etc., must be known to be independent of the conjunction of the remaining predicates. In this case the set of predicates  $p_1 \dots p_m$  will be called an independency.

A corresponding condition is required for the set of subjects  $s_1 \dots s_n$ ; i.e. there must be a predicate  $\pi_m$  such that

$$s_n \tilde{\chi} \pi_m \tilde{\chi} s_1 s_2 \dots s_{n-1}$$

as also a predicate  $\pi_{m-1}$  such that

$$s_{n-1} \tilde{\chi} \pi_{m-1} \tilde{\chi} s_1 s_2 \dots s_{m-2}$$

and so on. In other words, each additional subject  $s_n, s_{n-1}$ , etc., must differ in character in at least one predicate from the remaining subjects. When the set of subjects satisfies this condition, we shall speak of it as a variancy.

§ 7. Thus the parallel terms independency and variancy—the first applying to a collection of predicates, and the second to a collection of subjects—can be defined absolutely; i.e. without reference to any other named predicate or subject. Now such an independency must be distinguished from another set of predicates which are invariably found to be concomitant with the independent set in our instantial evidence. If  $p$  typifies such a concomitant predicate,  $p$  will differ from any of the predicates in the independent set in the point that there is no subject, such as  $\sigma$ , which is known to be characterised by the set  $p_1 \dots p_m$ , which is known not to be characterised by  $p$ . Again the set of subjects  $s_1 \dots s_n$ , which constitutes a variancy, must be distin-

guished from another set of subjects which are found to be congruent with the set  $s_1 \dots s_n$  in all their known common characters. If  $s$  typifies such a congruent subject,  $s$  will differ from any of the subjects in the variant set in the point that there is no predicate such as  $\pi$  which is known to characterise the set  $s_1 \dots s_n$ , which is known not to characterise  $s$ . There are thus two complementary aspects of our evidence, which have been summed up in the negative form (1) that no subject known to be characterised by  $p_1 \dots p_m$  is known not to be characterised by  $p$ ; and (2) that no predicate known to characterise  $s_1 \dots s_n$  is known not to characterise  $s$ . These two negatively formulated conditions may be otherwise expressed affirmatively; viz. first that every subject that is known to be characterised by  $p_1 \dots p_m$  is also characterised by  $p$ ; and secondly that every predicate that is known to characterise  $s_1 \dots s_n$ , also characterises  $s$ . These conditions in our ascertained knowledge are equivalent to the statement that all our evidence is unexceptionally favourable towards the conclusion  $s$  is  $p$ ; i.e. the evidence comes under our first heading (a). Where  $s$  is unexceptionally congruent with the collection  $s_1 \dots s_n$  as far as our knowledge reaches, we are led to the inference that it will be congruent with this set in all other unknown as well as known predicates. Again where  $p$  is unexceptionally concomitant with the set  $p_1 \dots p_m$  as far as our knowledge reaches, we are led to the inference that it will be concomitant with this set in all other unknown as well as known subjects.

§ 8. Having brought out the possible parallels between substantives and adjectives, or subjects and predicates, as far as analogy permits, we have arrived at

a point where an irreducible contrast between these two categories prevents the further formulation of the intensive and extensive aspects of eduction on precisely the same lines. The substantive or subject of the simple proposition ' $s$  is  $p$ ' is ultimately identified and distinguished from the adjective or predicate by a definite type of act of thought. (The subject  $s$  is definitive and nameable, as well as identifiable and distinguishable by such extrinsic relations as temporal and spatial position; such identification may therefore be said to be determined by an act of separation.) On the other hand the predicate  $p$  stands for an adjective or character predicable of different substantives by means of an act of comparison or discrimination; predication, therefore, may be said to be determined by an act of discrimination.

§ 9. My whole philosophical attitude depends upon the recognition of a fundamental distinction between these two types of acts, specially denominated separation and discrimination—a distinction which corresponds in my general view to that between the particular and the universal. Inasmuch as substantival separation is a necessary preparatory step to identification or comparison of character, separation may be said to be prior to discrimination; and this entails a further important contrast between subject and predicate. Thus there is, amongst substantives, nothing corresponding to the distinction and relation which obtains amongst adjectives between determinables and determinates; substantives proper, i.e. existents, are necessarily *distinct* just because they occupy positions in the *same* totality of time and space; whereas determinates which are *opposed*, are those which belong to the same determinable.



## CHAPTER V

### PLURALITY OF CAUSES AND OF EFFECTS

§ 1. IT has become common in modern logical works to deny the applicability of the plurality of causes where scientific analysis has succeeded in the final formulation of natural laws. While agreeing in the main with this logical position, I consider that the notion of plurality of causes and effects is applicable where such scientific analysis is incomplete, and that its discussion therefore has a place in the logical foundations of science. Since there are various senses in which the phrase is used, it will be convenient to arrange the discussion under a number of heads of discourse.

The most elementary notion of plurality of causes is that which Aristotle called 'the fallacy of the consequent,' meaning by this what now-a-days we call the fallacy of simply converting a universal affirmative. Thus, to infer from 'Every  $c$  is  $e$ ' that 'Every  $e$  is  $c$ ,' or from 'If  $c$  then  $e$ ' that 'If  $e$  then  $c$ ' is what in modern logic we designate as the confusion between a proposition and its complementary; and this fallacy has practical application where  $c_1$  (say) stands for the characterisation of a cause, and  $e_1$  for the characterisation of its effect. It is obviously fallacious to infer from the manifestation of the character  $e_1$  that the character  $c_1$  will have been manifested; for the effect  $e_1$  may have been due in any given instance to some other cause, say  $c_2$ , and this possibility entitles us to speak familiarly of plurality of

causes in cases where simple conversion is illegitimate. This first application of the conception may be expressed either as an *alternative* of predications, or as a *conjunction* of propositions. Thus, starting with the effect  $e_1$ , the proposition expressing plurality assumes the alternative form; 'Every  $e_1$  is either  $c_1$  or  $c_2$  or  $c_3$ , etc.'; but starting with the cause, it assumes the form of a conjunction of propositions, namely; 'Every  $c_1$  is  $e_1$ , and every  $c_2$  is  $e_1$ , and every  $c_3$  is  $e_1$ , etc.' It is usual in the general exposition of plurality of causes to employ the alternative form of the proposition; and this has led to the view that it is impossible to infer from the characterisation of an effect what in any instance may have been its specific cause; for the alternative form of proposition 'Every  $e_1$  is either  $c_1$  or  $c_2$  or  $c_3$ , etc.' renders it impossible when the effect is characterised merely by the character  $e_1$ , to determine amongst the several alternatives  $c_1, c_2, c_3$ , etc., which has actually operated in any given case. But there is nothing in the fact of such plurality of causes as this incompatible with the view that a sufficiently precise characterisation of the effect enables us to assign the specific cause in any given instance. For example, the alternative proposition can be consistently held along with the determinate propositions 'Every  $e_1 f_1$  is  $c_1$ ,' and 'Every  $e_1 f_2$  is  $c_2$ ,' and 'Every  $e_1 f_3$  is  $c_3$ ,' etc.; and it is important to note that this holdsequally of the sufficiently precise characterisation of the cause. It is a fundamental error common to most accounts of plurality to suppose that something is true of the relation of effect to cause, which is untrue, *mutatis mutandis*, of the relation of cause to effect. There is absolute reciprocity between cause and effect,

and insufficient determinateness in the assignment of either prohibits inference, whether from effect to cause, or from cause to effect. The apparent want of reciprocity is simply due to an imperfection of terminology. The word cause is understood to denote a *completed* assignment of cause-circumstances; while the term effect is used to denote an *incomplete* assignment of effect-circumstances. An incomplete assignment of cause does not enable us to infer a determinate effect, and hence, in this sense, it is not true that the same cause involves the same effect. On the other hand, a completed assignment of the effect does enable us to infer a determinate cause; and hence the statement that the same effect does not involve the same cause is equally false.

§ 2. The question which naturally next arises is what constitutes a complete assignment of cause or of effect, such that we are able to infer from one to the other? There is, in my view, no general answer to this question: each case must be treated *ad hoc*. Complete assignment of cause will be relative to the more or less arbitrary assignment of the effect-character; and where effects are assigned with perhaps equal degrees of indeterminateness, very different degrees of determinateness might have to be assigned to the cause in order to permit of inference. This holds equally with regard to the complete assignment of an effect. It will help to elucidate the problem to distinguish under the term 'cause,' the 'completed cause' from any 'cause-factor'; and, under the term 'effect,' the 'completed effect' from any 'effect-factor.' Thus, relatively to any effect-character  $e$ , we shall speak of the conjunction  $abc$  as constituting the completed cause, when the universal

proposition 'Every  $abc$  is  $e$ ' holds. In this case, the several characters  $a, b, c$ , are cause-factors of the given effect characterised as  $e$ ; and the truth in the doctrine of plurality of effects is expressed in the statement that a recurrence of  $a$  alone or of  $b$  alone or of  $c$  alone does not entail a recurrence of any the same effect-character such as  $e$ ; in other words, the mere recurrence of a cause-factor does not ensure the recurrence of any the same effect-character. When then we speak of a completed assignment of the cause, we mean simply such an assignment as will ensure identity, in its several recurrent manifestations, of some character in the effect. Similarly in the case of effect: here we start with some cause-character  $a$ , and proceed to establish some conjunction of effect-characters, say  $pqr$ , such that wherever the conjunction  $pqr$  is manifested, we can infer the causal operation of  $a$ ; the conjunction  $pqr$  is then denominated the 'completed effect' of  $a$ . On the other hand, the several characters  $p, q, r$ , are effect-factors from neither of which alone could the operation of  $a$  have been inferred. This last fact expresses the truth inherent in the doctrine of plurality of causes. From the effect-factor  $p$  alone, or  $q$  alone, or  $r$  alone, we could not have inferred any such determinate causal factor as  $a$ ; while from the conjunction of these characters, which we have called the completed effect, the character  $a$  of the cause may be safely inferred.

§ 3. To bring out the fundamental principle that when any invariability of relation can be established, it is always a determinate conjunction of characters from which some other character can be inferred, we will symbolise the completed cause of  $p$  as  $abcd$  where

'Every  $abcd$  is  $p$ ,' and the completed effect of  $a$  as  $pqr$  where 'Every  $pqr$  is  $a$ .' We do not expect in general to be able simply to convert these uniformities in the forms 'Every  $p$  is  $abcd$ ' or 'Every  $a$  is  $pqr$ '; nor, from the universal 'Every  $abcd$  is  $p$ ' do we expect to be able to infer that every  $p$  is  $a$ , or that every  $p$  is  $b$ ; or from the universal 'Every  $pqr$  is  $a$ ' that every  $a$  is  $p$ , or every  $a$  is  $q$ , etc.; for any given case of  $p$  might have been  $a'b'cd$ , and any case of  $a$  might have been  $p'q'r$ . So from any one effect-character  $p$  we should not in general be able to infer any one of the cause-factors which together constitute its completed cause; and from any one cause-character  $a$ , we should not in general be able to infer any one of the effect-characters which together constitute its completed effect.

For brevity's sake we will here replace the compound symbol  $abcd$  by the letter  $x$ , and the compound symbol  $pqr$  by the letter  $z$ . We then have the two universal propositions 'Every  $x$  is  $p$ ' (where  $p$  is an effect-character, and  $x$  its completed cause), and 'Every  $z$  is  $a$ ' (where  $a$  is a cause-factor, and  $z$  its completed effect). As we have already pointed out, the doctrine of plurality may be expressed in two ways—either in the alternative form of predication, or in the conjunctive form; and so the above discussion may be summarised either in the alternative form 'Every  $p$  is  $x$ , or  $x'$ , or  $x''$ , etc.' and 'Every  $a$  is  $z$ , or  $z'$ , or  $z''$ , etc.' or in the conjunctive form 'Every  $x$  is  $p$ , and every  $x'$  is  $p$ , and every  $x''$  is  $p$ , etc.' and 'Every  $z$  is  $a$ , and every  $z'$  is  $a$ , and every  $z''$  is  $a$ , etc.' Colloquially expressed these propositions are embodied in the statements that 'in different instances *different* causes point to the same effect'; and that 'in

different instances *different* effects point to the same cause.' Our immediate problem, then, is to examine further into the significance of the term 'different,' and to enquire into its adequacy for expressing the true meaning of plurality.

§ 4. When we assert that under different antecedent conditions the same consequent is manifested, we may be understood to be simply asserting that any such proposition as 'Every  $x$  is  $p$ ' means that  $p$  can be predicated in all cases—differing in an indefinite number of characteristics from instance to instance—where the character  $x$  is manifested. But the mere difference of circumstances in such cases, being followed by the same effect, cannot be said to constitute plurality of causes; for, if a character, expressed by the determinable  $V$  say, varies quite indefinitely without affecting the character  $p$ , then the various values of  $V$  which may be manifested from instance to instance are simply eliminated in the proposition that 'Every  $x$  is  $p$ .' In order to speak properly of plurality, the case must be such that *some* but not *all* of the possible values of a determinable can be substituted in the universal proposition; for instance where,  $X$  being the determinable, 'Every  $x$  is  $p$ ,' 'Every  $x'$  is  $p$ ,' 'Every  $x''$  is  $p$ ,' are the only three values of  $X$  which yield  $p$ . The conception of plurality, therefore, requires us to refer to a restricted range of alternatives; some, but not all, values of  $X$  point to  $p$ ; some, but not all, values of  $Z$  point to  $a$ . Further  $x$ ,  $x'$ ,  $x''$ , it will be observed, are contraries, opponents or disjuncts, and so the term 'different' in this account has an added significance. A rough example will illustrate this point: death is sometimes caused by poison, sometimes by a

blow on the head; the mere *difference* between these two causes does not bring out the full significance of plurality, for poison without the blow, or the blow without poison, would produce the same effect death, and these are contrary circumstances. A similarly rough illustration of plurality of effects might be taken from the fact that both a picture and a pattern point, as effects, to human action as cause; here again the full significance of plurality is brought out not merely by the difference between pattern and picture, but by the contrariety of these effects, one being a picture which is not a pattern, and the other a pattern which is not a picture; while both opposed effects point to the same cause, namely human purpose or agency. In both these rough illustrations where contrariety or opponency are substituted for mere difference, we must emphasise the further consideration that the range of opponency which gives significance to the idea of plurality is restricted, and is therefore to be contrasted with a completely unrestricted range where possible variable values would be simply eliminated. It is not every splash of colour that points to human purpose, but, in our assumed example, only those which are pictorial or symmetrical; and it is not every condition of the head or inner organs that would point to death, but only some selected and definable conditions.

§ 5. As stated in the previous section, each of a number of opponent cause-characters  $x, x', x''$  (say) entails the same effect-character  $p$ , when conjoined with other cause-characters respectively,  $y, y', y'' \dots$  (say),  $X$  and  $Y$  being determinables indefinitely variable. But it must also be recognised that each of a certain *finite* set of values  $x_1, x_2, x_3$  (say) of  $X$ , when conjoined with the *same* value

$y$  of  $Y$  may entail the same value  $p$  of the effect-character  $P$ . This fact is best illustrated by a graph in which the abscissa represents the variable cause-character, and the ordinate the variable effect-factor, and where—other relevant circumstances being unchanged—a horizontal line meets the graph in two or more points  $A, B, C, D \dots$ . If one observation is represented at one point  $A$ , there is a chance—which however is small—that the second observation should be on another point  $B$  where the horizontal through  $A$  meets the curve; but this initially small probability decreases continuously with every fresh instance. This familiar fact requires us to modify somewhat our formulation of the figures of Demonstrative Induction. Thus, as regards the figure of Agreement, it was said that if any two values, say  $a_1$  and  $a_2$ , of the cause-factor  $A$  (all other factors,  $b, c, d, e$  remaining constant) entail the same effect value  $p$ , then  $P$  is independent of  $A$  (under the circumstances  $bcd e$ ). But such independence would be falsely inferred if  $a_1$  and  $a_2$  were two values which yielded the same effect  $p$ . The probability, under the ordinary circumstances of experiment and observation, of precisely these two values occurring is in general so small as to be negligible; and in order to diminish even this small probability more than two values of  $A$  should be experimentally instanced. A similar correction is required for the figure of Difference. Thus, if  $d$  and  $d'$  have yielded different values  $p$  and  $p'$  of  $P$  (under otherwise constant circumstances) we cannot infer that literally *any* other value of  $D$  would yield a different value of  $P$ ; for there is a small chance that in another instance we might happen to hit upon one of the values of  $D$  that yields the same value  $p$  of  $P$  as that yielded by  $d$ .

§ 6. We have attempted to prove that the relation between cause and effect is reciprocal in the general sense that whatever is true of cause to effect, will be true of effect to cause, whether the relation asserted is in the form of a universal, a particular, a conjunctive, or an alternative proposition. We now proceed to investigate how the relation of cause to effect can be rendered reciprocal in the more precise sense in which it involves the conjunction of the two complementary universal propositions 'Every  $C$  is  $E$ ' and 'Every  $E$  is  $C$ .' Now in formulating any universal proposition, we begin by taking as predicate term an arbitrarily assigned character which may stand either for a cause-character or for an effect-character. We then attempt to find some conjunction of characteristics from the manifestation of which the arbitrarily assigned character may be inferred. This conjunction of characteristics is denominated 'the completed cause' when the original character stands for an effect; and is denominated the 'completed effect' when the original character stands for a cause. Thus, in general, starting with the predicate term  $p$ , we attempt to establish such a proposition as 'Every  $abcd$  is  $p$ .' If, at this point, we turn from  $p$  to  $abcd$ , and attempt to find a universal mark from which the conjunct character  $abcd$  could be inferred, we discover a mark definable by some such conjunction as  $pqr$ . We should then have the two universals 'Every  $abcd$  is  $p$ ' and 'Every  $pqr$  is  $abcd$ ,' where  $abcd$  defines the completed cause of  $p$ , and  $pqr$  the completed effect of  $abcd$ . The next step in our approximation to a reciprocal universal is to attempt to find the completed cause of  $pqr$ . Since  $abcd$  constitutes the

completed cause of  $p$ , the completed cause of  $pqr$  will include  $abcd$ , but it may require further determination:  $abcdef$ , let us suppose, is found to be the completed cause of  $pqr$ . By this process we approximate more and more closely to a reciprocal universal, which may ultimately be supposed to assume the form 'Every  $abcdef$  is  $pqr$ , and every  $pqr$  is  $abcdef$ .' Let us retrace the steps by which we arrive at this double formula: beginning with  $p$  alone, we established 'Every  $abcd$  is  $p$ '; next with the conjunction  $abcd$  as our starting-point, we established the universal 'Every  $pqr$  is  $abcd$ '; then with the conjunction  $pqr$  as our starting-point we established the universal 'Every  $abcdef$  is  $pqr$ '; and lastly, starting with the conjunction  $abcdef$  we established the universal 'Every  $pqr$  is  $abcdef$ .' This procedure is assumed to have reached its termination from the fact that  $abcdef$  implies not only  $pqr$ , but also  $s$ , so that the relation of inferability between  $abcdef$  on the one hand and  $pqr$  on the other hand is reciprocal.

§ 7. The notion of a completed effect or a completed cause may be approached from another point of view. Taking as before  $abcd$  to be typical of a cause-conjunction, we shall enquire what effect-characters can be inferred wherever this cause-conjunction is manifested. Let us suppose that  $p$ ,  $q$ , and  $r$  are three independently definable characters which can be called effects of the junction  $abcd$ , so that the universal proposition 'Every  $abcd$  is  $pqr$ ' can be asserted. But if  $p$ ,  $q$ ,  $r$  are to be called effects proper to the cause-conjunction  $abcd$ , a further condition beyond the truth of the universal proposition is required. Not only must it be true that 'Every  $abcd$  is  $pqr$ ,' but it must also be true that neither



$p$ , nor  $q$ , nor  $r$  could be inferred as effect from any cause-conjunction involving some only of the factors  $a, b, c, d$ . For example, if every  $abc$  were  $p$ , then  $p$  would be not an effect proper to  $abcd$ , because  $p$  would be the effect proper of the part-conjunction  $abc$ . Thus in order to find the effects proper to the conjunction  $abcd$ , we must exclude all effects which could be inferred from  $a$  alone, or from  $ab$  alone, or from  $bc$  alone, or from  $abc$  alone, etc. Or, to take another illustration of a closely connected point, the universal proposition 'Every  $abcd$  is  $x$ ' would not be a true expression of causal law supposing that we could have dropped the  $d$ , and expressed the universal in the wider form 'Every  $abc$  is  $x$ .' Relatively to  $x$ , the conjunction  $abcd$  would contain a superfluous factor, and super-complete assignment of cause is as invalid as insufficient assignment; thus, in formulating a universal proposition stating a causal relation, the cause must not only be complete, but it must *not* be super-complete. These conditions are effected, in the case before us for instance, by taking separately each of the cause-characters  $a, b, c$  and  $d$ , and finding the effects which are due first to the factors taken one by one; secondly to the factors taken two by two; and thirdly to the factors taken three by three; and thus finally to reserve as the effects proper to the conjunction  $abcd$  those which can be inferred from the complete conjunction alone. Assuming then that we have standardised our causal formula by excluding those effects for which  $abcd$  would be a super-completed cause, we shall suppose that  $p, q, r$  severally are effects proper to the completed cause  $abcd$ . The question next arises as to whether  $pqr$  is the completed effect of  $abc$ ; for just as

in completing the assignment of cause we have to avoid the error of redundancy, so also—though for a different reason—we have to avoid redundancy in our assignment of the completed effect. The super-completed effect will be one in which we have failed to distinguish an effect from the effect of an effect; thus we should have wrongly assigned  $pqr$  as the effect of  $abcd$ , if  $r$  itself were the effect of  $pq$ . In other words, the completed effect must consist in an independent conjunction. Summarising the conditions, then, for the correct formulation of the causal law which presents  $abcd$  as the cause of  $pqr$ :

(a) The characters  $a, b, c, d$  must be independently definable and independently co-variable.

(b) The characters  $p, q, r$  must also be independently definable and co-variable.

(c) None of the effects  $p, q, r$  must be inferable from a conjunction included in, but less comprehensive than, the conjunction  $abcd$ ; and conversely.

If  $p, q, r$  are the only effect-characters which satisfy these conditions, then the conjunction  $pqr$  may be called the completed effect of the cause-conjunction  $abcd$ . Any other effect-character such as  $s$  would have to be excluded, either because it was an effect of  $pqr$ , or because it was an effect of a conjunction more comprehensive or less comprehensive than  $abcd$ . Finally then, when the above conditions are satisfied the relation between the cause-conjunction  $abcd$  and the effect-conjunction  $pqr$  is reciprocal; so that 'Every  $abcd$  is  $pqr$ ' and 'Every  $pqr$  is  $abcd$ '; moreover, both uniqueness of effect entailed by the given cause, and uniqueness of cause entailed by the given effect are secured.

## CHAPTER VI

### CAUSE-FACTORS

§ 1. THE validity of the antithesis between nomic necessity and universality of fact being admitted, it has frequently been supposed that, within the range of the nomically necessary, causal laws can be distinguished from non-causal laws. But this view must be rejected. Causal laws have been held to apply only where *change* is involved; we have therefore to enquire into the significance of this notion, and in place of the somewhat obscure term *change*, I shall introduce the notion of alterable as opposed to unalterable states of a thing. This phraseology would not be admitted by those philosophers who recognise only events or occurrents, and do not allow, except for linguistic convenience, the notion of a continually existing thing to which states or occurrents are referable. I must here restate in more detail my view that any occurrent is to be referred to a continuant, and that the relation of an occurrent to its continuant, or inversely of a continuant to any of its several occurrents, is a unique relation, to which there is no analogue in any other aspect of reality. A relation sometimes hastily confounded with that of occurrent to continuant—which I will call *inherence*—is the relation of substantive to adjective, which I call *characterisation*. But since an occurrent may be variously characterised it is obvious that it stands to its characterisation as substantive to adjective; the relation,

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therefore, of the occurrent to its characterisations cannot be identified with the relation in which it stands to its continuant. The continuant itself might be called a substantive proper, in the narrowest possible sense of this phrase; but I include under the phrase substantive proper both the occurrent and the continuant, thereby indicating that the relation of the one to the other is not the same as that of substantive to adjective.

To define more explicitly the notion of a continuant, we will assume that any continuant has several modes of existence, or rather modes of manifestation of existence, each of which may theoretically be conceived as a determinable<sup>1</sup>; and according to the nature of this set of determinables, the continuant may be said to belong to one or another category. We assume further that during the period throughout which a continuant exists, every one of its modes is being manifested in some or other of its determinate forms. In the proper mathematical sense, time is of one dimension, but in order to conceive of the existence of a single continuant, it will be helpful to represent time, in a sort of figurative imagery, as having a number of parallel dimensions. Applying this figure of speech to the continuant, we may say that its existence is prolonged along a number of parallel lines of time, each of which manifests from moment to moment the several modes of manifestation in one or other determinate form.

<sup>1</sup> My terminology should be compared specially with that of Descartes and Spinoza. What I call a determinable is almost equivalent to what they call an attribute, and my determinate almost equivalent to their mode of an attribute. My use of the term 'mode' will, therefore, be seen to differ from theirs.

These lines of time, therefore, are conceived as being completely filled or occupied by actual manifestations; and the conception of parallel time-lines must be extended so as to apply to all continuants.

§ 2. With these preliminary remarks I will pass to the temporal and spatial relations involved in the conception of causality. In the first place the antithesis between occurrent and continuant corresponds to the antithesis between the transient and the permanent or persistent. Popularly speaking, what exists may have only transient existence, or else persist continuously throughout a period of time, perhaps indefinitely prolonged at both ends. In attributing continued existence to a thing, we do not mean that some property of the thing continues unchanged; for a property stands to its continuant in the relation of adjective to substantive. There is a further distinction amongst properties which characterise a continuant, according as these change or persist unchanged throughout a period of time. The continuity of the existent is something behind even the possibly changing properties, and change applies not to the continuant itself, but to the adjectives which characterise it or its occurrents. Often the term cause is applied indiscriminately either to the continuant itself or to some of its properties regarded as permanent in relation to the particular occurrences or events as effects. Cause, in this sense, is essentially something persisting throughout time, and effect something essentially transient and alterable; so that the cause is not homogeneous with the effect, and this usage of the term cause is to be carefully distinguished from the notion as applied to related occurrences. Mill fails to point out this dis-

tinction when he allows himself to deal in a separate chapter with permanent causes or cause-agents, and in so doing departs entirely from his preliminary account of cause and effect as temporally related, the one as antecedent and the other subsequent in time. We must be on our guard, then, against the habit of confusing causality regarded as a relation between events with causality regarded as the relation of a permanent existent to its alterable conditions or relations. For the present I propose to confine the discussion to the more common and familiar application of this notion to occurrences. In this sense cause and effect are homogeneous; i.e. the same sort of thing that can be said about the relation of cause to effect can also be said about the relation of effect to cause. Thus if a cause process is simultaneous with an effect process, this temporal relation of simultaneity is convertible; or again if the cause process is anterior to the effect process, the latter is posterior to the former. There is a further reciprocity between cause and effect when we conceive of objective determination in its wide sense; for it is held in modern times that the specific characterisation of an effect determines the cause in the same objective sense as the specific characterisation of a cause determines the effect. This view is almost universally accepted, at any rate from the epistemic point of view; i.e. it is held that the knowledge, say, of a sufficiently omniscient being of what is customarily called the effect, would permit of inference as to the nature of the cause with just as much certainty as inference from the knowledge of the cause to the knowledge of the effect. If this be so a real problem arises as to whether

ontologically, as opposed to epistemologically there is any objective antithesis between the relation of cause to effect and that of effect to cause, since each of them may be said to determine unequivocally the nature of the other.

These general considerations lead to an apparent paradox with respect to the reference of causality to time and space. Philosophers, scientists and logicians alike have often put forward as the one supreme principle of causality, that the causal dependence of event upon event is wholly unaffected by temporal and spatial differences. On the other hand the analysis of every phenomenon in terms of cause and effect assigns spatio-temporal relations between cause and effect. This paradox is removed by considering that the formula in accordance with which one event is causally connected with another, is independent of the date and location of the events, but dependent on the temporal and spatial relations between them.

§ 3. The alleged distinction between two types of objective law serves to introduce Mill's distinction between uniformities of co-existence and causal laws. The phrase 'uniformity of co-existence' requires special consideration, because it has to be distinguished on the one hand from formal universals, and on the other from causal laws. Formal universals are concerned with the spatial and spatio-temporal relations involved in the notion of movement. The difference between such formulae and those which connect the properties of continuants or the characters of occurrences, is that the latter refer to existents whereas the former do not; the term existent here being understood to apply to what is potentially

or actually manifested in time, or in space, or in both time and space. Briefly the geometrical and kinematic formulae comprised under formal universals express the nature of time and space themselves; whereas uniformities of co-existence and of causation express the nature of that which occupies time and space. The latter uniformities therefore include or presuppose the former, while obviously the former do not include or presuppose the latter. Passing now to the contrast or connection between uniformities of co-existence and uniformities of causation, the two points which we shall proceed to maintain are, first, that no causal law can be formulated except by reference to co-existing properties of continuants as well as by reference to changeable occurrences; and secondly, that the required distinction is not simply one of temporal relation, such as simultaneity and sequence.

§ 4. If we consider what is involved in defining or describing an occurrence, we find that it must always entail reference to a continuant; and that one occurrence is defined as agreeing with or differing from another, by reference to the *properties* of the continuants concerned. For example, the occurrence described as drinking water is different from the occurrence defined as drinking ether, not by reference to anything which could be described in terms of actual perceptible phenomena, but by reference to the different properties or potentialities implied by the terms ether and water respectively, which denote different *kinds* of continuants. It is true that the smell and taste of ether would immediately distinguish it in sensation from water; but for a person who might accidentally have lost his susceptibility to

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smell and taste these perceptible differences would be unnoticed. Hence in considering the causal conditions which produce the different effects following upon the taking of ether or the taking of water, the different *properties* of these substances must be specified. Further proof of the inadequacy of the statement of causation which regards the cause as an actual occurrence related as simultaneous with or antecedent to the effect occurrence, lies in the fact that the cause assigned to account for a given effect includes not merely what has occurred in actuality, but what would have occurred under totally different circumstances. Thus the cause assigned to account for the observed effects of drinking ether would be that ether is poisonous, and this statement, though explicitly asserting the co-existence of certain properties, is implicitly a statement of causal law, presumably discovered by means of experiments in the laboratory.

§ 5. Not only is the *cause*, in my view, something more than a mere actual occurrence, but the *effect* may be something more than a mere actual occurrence; for there are many cases in which a complete account of the effect must include besides what could be characterised in an occurrence as immediate and actual, a characterisable change of property, i.e. of a potentiality that may be actualised in future manifestations. This is most obviously illustrated from the phenomena of habit and memory; for it is in mind that modifiability of property is specially prominent. Another respect in which such phenomena differ from those which are described in an ordinary account of an occurrence is that whereas an occurrence, taken as effect, is generally referred to a

single completely characterised occurrence as cause, in the case of habit, the effect produced must be accounted for, not by a single previous occurrence, but by a repetition of occurrences agreeing with one another in some respect<sup>1</sup>. These illustrations suffice to show first, the necessity of referring an occurrence to a continuant; and secondly, the necessity of including in an account of causal conditions, properties defining the *potentialities* of occurrents, as well as characters describing the actual occurrence. My terminology may be compared with the Aristotelean classification of causes; for Aristotle's material cause corresponds closely to what I call the continuant whose nature is manifested in causal processes; and his efficient cause approximately corresponds to what I call the property, and which, when analysed as a potentiality corresponds to the Greek term *δύναμις*. Finally what the scholastic logicians term the 'occasional cause' is to be understood as equivalent to the occurrent cause. It should be noted, however, that such distinctions are incorrectly described as distinctions amongst causes, for they are really distinctions amongst causal factors. Thus the continuant, the property of the continuant, and the occurrence, are three factors which jointly constitute the completed account of the cause. When a causal law is expressed in condensed form as a coexistence of properties, it is absolutely essential that the term used to denote the cause should not connote a property which represents the effect, for otherwise the supposed law is nothing but a

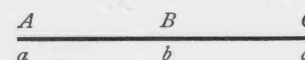
<sup>1</sup> This may of course be resolved into the preceding; for each single occurrence effects a change in the potentialities of future occurrences.



verbal proposition. For example, the proposition 'Poison always kills people' appears to express a uniformity of co-existence between the property defined as poisonous, and the property of occasioning death whenever introduced into a living person; but since this last property merely gives the meaning of the first, the proposition neither expresses a genuine law of causation nor a genuine law of co-existence.

§ 6. Turning now to our second point of distinction and connection between uniformities of co-existence and uniformities of causation, we must direct special attention to the occurrent factors in causation, with the object of examining first the temporal and later the spatial relations between the cause-occurrence and the effect-occurrence. The typical case of causation which has figured most prominently in philosophy since the days of Hume, is where the cause-occurrence is taken to have preceded in time the effect-occurrence. The language used to express the relation of temporal sequence or succession has generally suggested the idea that the cause-occurrence can be dated at one moment of time, and the effect-occurrence at another moment of time, with a temporal interval between the two moments. Philosophical criticism has generally rejected this account on the ground that it implies, as a necessary condition for the existence of the effect, the *non*-existence of the cause. In other words, the time at which the cause, as here regarded, operates in determining the character and existence of the effect, is the time at which the cause has ceased to exist, and can therefore no longer manifest its character. This attack upon the common statement of causation has led to an attempt

to overthrow the notion of causality itself, on the ground that it involves an irremovable paradox or contradiction. Now we have only to apply the above account of causation to the simplest known case of causal process to see that in truth it is fallacious. The mere datum which defines the collocation of a system of particles, would not enable us, even with the completest knowledge of the causal laws of motion, to assign their subsequent positions. The datum in this case defines an occurrence by the position at a moment of time of each of the particles; but the further datum required, in order to ascertain the positions at a subsequent moment of time, is the rate of movement of each particle within a period of time. This simple case points to the general principle for defining the temporal relation between a cause-occurrence and its effect-occurrence. Instead of dating a cause-occurrence and an effect-occurrence *at* two separated moments of time, we must define the cause-occurrence as a process going on *within* a certain period of time, and the effect-occurrence also as a process going on within a certain period of time. If the period assigned to the cause is earlier than, and not simultaneous with, that assigned to the effect, then the two periods must ultimately be taken as strictly contiguous: that is, the terminal phase of the cause-process coincides in time with the initial phase of the effect-process.



The line drawn above will serve to represent the difference between the inadequate and the adequate account of cause and effect. If an occurrence *A* be dated at the

moment  $a$ , and an occurrence  $C$  at the moment  $c$ , and  $A$  be then defined as the cause of the effect  $C$ , the account is inadequate; for not only does it involve the above-mentioned paradox, that the non-existence of  $A$  is a condition for the existence of  $C$ , but the account fails to assign any principle for determining the interval of time which must elapse between the moment  $a$  at which  $A$  has ceased to be, and the moment  $c$  at which  $C$  is manifested. If the time-interval between  $A$  and  $C$  is phenomenally unfilled, no account can be given of its length; we must therefore represent the interval between  $a$  and  $c$  as occupied by a process of change, say from phase  $A$  at moment  $a$  to phase  $B$  at moment  $b$ , and again from this latter to phase  $C$  at moment  $c$ . We shall then no longer speak of phase  $A$  at moment  $a$  as the cause of phase  $C$  at moment  $c$ , but rather of the change from  $A$  to  $B$  within the period of time  $ab$  as cause of the change from  $B$  to  $C$  within the period of time  $bc$ ; where no empty gap of time separates the cause from the effect. The cause in this case may still be said to precede the effect, but it is necessary to add that the temporal relation is one of strict contiguity. When we can quantify the differences of phase, it is possible not only to indicate the nature of the change which takes place within the whole period, but to correlate the degree of change from  $A$  to  $B$ , and  $B$  to  $C$ , with the time-relations  $ab$  and  $bc$ . In the simplest case, the quantity or degree of change is proportional to the period of time within which the change takes place: for instance, to illustrate the first law of motion,  $A, B, C$  might stand for successively occupied positions of a moving particle, so that  $AB$  and  $BC$  represent distances;

then the distances  $AB, BC$ , would be proportional to the periods  $ab, bc$ . In this way a principle is supplied to account for the length of time which must elapse between the occurrence of the cause and the occurrence of the effect, when these occurrences are dated at separated moments of time; and the initial paradox is removed.

## CHAPTER VII

### THE CONTINUANT

§ 1. WE have found in our analysis of the nature and determination of occurrences that some link, besides mere temporal and spatial connection, must exist between one occurrence and another in order that the first may be conceived as determinative of the second. This link I have called the substantive continuant, and in this chapter we shall examine the notion in detail, and show how it differs from the traditional conception of substance.

The simplest and most obvious illustration of the continuant is the case of the moving particle: thus, if two movements, defined in character by direction and velocity, and defined also by reference to the period of time and region of space within which each takes place, are to be conceived as connected, in the sense that the character, date and location of the one is determinative of the character, date and location of the other, then such a connection can only be presumed if the same material continuant is existentially manifested in the two movements. Apart from the introduction of the continuant, this simple example serves to illustrate the way in which identity and difference is involved in causality. We speak of two movements, and briefly call the one cause and the other effect. Inasmuch as the movements are two, they cannot be identical; so that it may be laid down as the first and most indubitable

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principle of causality that, whatever other subtle relations there may be between cause and effect, the relation of non-identity is to be unequivocally asserted. Hence, before the movements in question are connected as cause and effect, they must first be distinguished as *other* or *two*; and since time and place are the only conditions of otherness which have been conceived by the human mind in regard to physical phenomena, the movements in order to be conceived as two, must occupy either different periods of time and the same region of space, or different regions of space and the same period of time, or different periods of time and different regions of space. We will suppose that the two movements, connected as cause and effect, are referred to different periods of time and to different regions of space, and proceed to consider their characterisation as regards direction and velocity. In the very simplest case afforded by science of causal relation between movements, the direction and velocity of the movement called cause is identical with the direction and velocity of the movement called effect; in this case, therefore, cause and effect are non-identical as regards temporal and spatial reference, but identical as regards characterisation. Turning now from the adjectival characterisation of the occurrences to their substantival connection, our illustration may be expressed in terms of the first law of motion, as follows:— So far as the movement of a particle within one period of time is causally determinative of its movements within another period of time, the direction and velocity of movement is the same within these two periods. Here the two movements which are causally connected are movements of *one and the same particle*; so that

substantival identity is a notion essential to the understanding of the formula. It is to this substantival identity that I refer when I speak of a continuant.

§ 2. For logical purposes it replaces the term substance, familiar in metaphysics; but the various unfounded or *a priori* characteristics which philosophers have attributed to substance must be carefully separated from the essential logical residuum, and rejected from the notion of the continuant. Thus, in the first place, the conception of continuance has been extended into the infinite future and the infinite past. In my view, on the other hand, the application of continuance must be strictly limited to the periods of time in reference to which we can speak of change; that is, so far as we are justified in speaking of a state or condition as changing when we pass in thought from one period of time to another, so far are we justified in conceiving of the same entity or continuant as preserving its existence throughout the two given periods. This does not warrant us in asserting its existence either *before* or *after* these two periods. In physics, it is true that scientists have found it convenient to postulate an indefinitely prolonged existence into the past and future of the ultimate atoms which constitute matter; but this has no general logical or philosophical warrant, any more than there is philosophical or logical warrant for immortality.

§ 3. The next way in which metaphysicians have characterised any continuant or substance in an unwarrantable fashion, is by maintaining that amid all the alterations of state or condition which the substance undergoes, there are some one or more characters which

continue to be manifested unchanged. This position may be held in a more or less crude form. From my point of view, what is important to point out, is merely that substantival continuance does not necessarily imply any adjectival changelessness. When philosophers like Locke and Hume sought for significance in the conception of *identity*, as substantive continuance used to be called, they were continually guilty of confusing continued existence of the same substantive entity with qualitative or adjectival identity of character or state. Failing to find this qualitative identity, Hume explicitly rejected substantival identity; and those who opposed Hume held equally with him that substantive identity could not be maintained except in so far as qualitative identity could be established. I repeat then, that the conception of substantive continuance does not by itself carry with it the implication of unchanged character through the period of time to which the substantive continuance applies. As in the matter of absolute temporal permanence, so also in this question of unchanged character, the physicists have found it convenient to postulate in various forms an unchanged continuance of character in the atoms or compound bodies which constitute the matter of the universe. Though I have here called these postulates of the physicist assumptions, I do not wish to deny that some of them may have inductive warrant; to this we shall have to return when we consider scientific induction in detail.

§ 4. The third and last *a priori* attitude towards the notion of the continuant must be briefly treated. This is the contention that the ultimate continuant is simple and not compound. On this subject it is perhaps of

greatest importance at the present day to distinguish between compound, in the sense of a whole consisting of parts, and compound in the sense of involving inner causal or dynamic interaction. The former conception raises no serious problem, the continued identity of the whole being obviously involved in the continued identities of the parts. It is possible however to conceive of a compound entity which continues to preserve its identity through change of time, although none of the parts, which appear from time to time to constitute the whole, can be said to preserve their several identities. This may conceivably be explained by exhibiting a law or principle in accordance with which the compound continuant develops a changing character by means of the instrumentality of the dynamic interactions amongst the parts or components which from time to time constitute so to speak the substantial material of which the compound continuant is composed. Thus the law or principle according to which the character of the continuant at one time can be exhibited as depending upon its character at another time, may be the ground for asserting continued existential identity, although the material components of this continuant are not themselves continuant.

§ 5. We began our exposition of the continuant by an illustration from physical science, showing how the physical continuant is involved in the simple formula known as the first law of motion. We shall now bring forward an illustration of approximately equal simplicity from the psychical sphere. In the physical illustration we included reference to space as well as to time; in our psychical illustration we shall drop, for the present, any reference to space. If a sensation characterised in some

way, and a thought process, also characterised in some way, occur one within some period of time, and the other within the same or a different period of time, then the character and date of the sensation can only be conceived as determinative of the character and date of the thought process if, in the simplest case, the same psychical continuant is existentially manifested both in the sensation and in the thought process. Precisely as in the case of physical phenomena, change or alteration in time does not mean the replacement of a sense-experience of red, say, referred to one period of time, by a sense-experience of blue referred to another period of time; for the mere reference of differently characterised experiences to different periods of time does not constitute what we call change or alteration. Here as in the case of the physical continuant, we can only speak of change or alteration by conceiving of an existent which continues to exist within both the periods of time to which the change refers; and it is for this reason that we call such an existent a continuant.

But the notion of change, when applied to the psychical continuant, raises a peculiar problem when we consider the different kinds of experience referable to one and the same continuant: thus we may put the question whether it is correct to speak of a change of state when, for example, we refer a sensation to one date, and a thought or volition to another date; or when we refer say a colour sensation to one date and a sound sensation to another date. The mere fact that any colour sensation is by definition different from any sound sensation, and still more that any sensation is different from any thought or from any volition, does not appear to justify us in



speaking of change or alteration when such phases of experience are referred to different dates. On the other hand we should with less hesitation speak of change or alteration when the differing experiences come under the same determinable. A sensation of red followed by a sensation of blue—blue and red being determinates under the same determinable, colour—would appear to illustrate the notion of change of state more correctly than a sensation of red followed by a sensation of noise, or by a thought about geometrical relations, or again by a voluntary decision to get out of bed. Now the real reason why we apply the word change preferably to the first case and not to the second is because we suppose that the blue sensation has replaced the red sensation, so that at the time that the blue is manifested, the red has ceased to be manifested. It is thus the cessation of one character of our experience, and its replacement by another, that constitutes the essence of change.

§ 6. The above analysis helps towards a solution of the problem as to what it is that can be said to change. On the one hand, it cannot be the continuant itself, nor any of its properties, since these are asserted to be constant throughout the period of time to which the process of change is referred. Neither can it be the manifestations, dated at time-points, which can be said to change, since these merely replace one another from instant to instant. The clue to the problem is to be found in the theory of the determinable. The character of each dated manifestation is determinate, and a change implies always that the determinate character of the one manifestation at one instant is replaced at a sub-

sequent instant by a manifestation having a different determinate character under the same determinable. Thus we speak of temperature or colour or size or shape, etc., as changing or remaining constant during a certain period of time; it is therefore the manifestation—not of a *determinate*—but of a *determinable* that may be said to change. But further, the idea of change involves not only the adjectival determinable, but also the substantival determinandum; for change would have no meaning unless there were a continuant, which was necessarily manifested in a mode characterised by one or another determinate value of a determinable. Thus the substantival determinandum is conceived as continually manifesting one or another determinate character under the same determinable, and being potentially manifestable in a mode characterised by *any* value of the determinable. This aspect of the nature of change leads to the conception of that which determines this potentiality to become an actuality; in other words the conception of change brings with it the conception of causal determination.

To prevent one minor confusion it is necessary to point out that what holds of change proper holds also of the continuance of the manifestation unchanged; for the fact of continuance as well as of change requires the assignment of a cause. The fact that the popular mind demands only an explanation of change which will assign the event or occurrence operating as cause, is accounted for by familiarity with unchanged continuance in many manifestations. Actually the preceding unchanged continuance constitutes in such familiar cases the cause of the subsequent unchanged continuance; but it is only

when this continuance is interrupted that the question of the cause of interruption is generally raised. For this reason the conceptions of cause and of change are always supposed mutually to involve one another.

§ 7. The simple illustrations which we have brought forward of a physical continuant and a psychical continuant, have served to introduce the view that the two notions, familiarly known in philosophy as substance and causality, are mutually dependent the one upon the other. No adequate account of causality can be given without reference to the conception of substance, i.e. of an existent continuant, physical or psychical; and on the other hand, we can only assign properties to the substance or continuant by defining the modes according to which it is existentially manifested as a *causal* agent or re-agent. Thus what is called a property of a continuant is not an actually manifested character, but it defines what characters would be phenomenally manifested when certain assignable conditions occur. For example, the elasticity of an extensible string illustrates a property which we attribute to the string; it defines in general terms the degree of length which would be attained were the string exposed to a certain tensional force. A property, therefore, expresses a definable group of manifestations—not as actual—but as *potential*. The general formula for expressing the property of a continuant *c* assumes the shape: if a certain occurrence defined as *p* were to take place, in which the continuant *c* is patient, then a phenomenal manifestation defined as *q* would occur which is determined by the nature of *c*.

It should be observed that continuants—i.e. in ordinary language things—are classified according to their

properties; such familiar terms as solid, liquid and gaseous, for example, do not describe phenomena as actual but as potential, and they are typical of an innumerable host of terms in familiar use. It is not altogether easy to distinguish those adjectives in common use which denote respectively actualities and potentialities of manifestation; in fact the problem of their distinction raises points of philosophical importance. As regards *physical* continuants, in predicating such adjectives as those of position, shape or size, or more generally of spatial configuration, I shall assume that we are predicating actualities; but these are the only adjectives descriptive of merely physical phenomena which are regarded unanimously by physicists as actually manifested. I myself hold that there is, besides spatial configuration and motion, another physically actualised mode, namely force as defined in statics. Physicists appear to me to maintain that, where equilibrium exists, what has been called force is merely an indication of potentiality for movement; so that only the energy of movement is *actual*, and in static condition force is held to be a myth. But in my view, static force represents a real condition of a body; e.g. when a heavy body is in equilibrium on a horizontal surface, the force called pressure actually exists, and is not a mere measure of what would take place if free motion were permitted. One evidence for this view is the recognised association of strain with stress; i.e. stress is a particular example of force which is correlated with strain; this latter being a geometrical conception. The formula according to which strain and stress are mutually connected and yet distinguished, so that they stand to one another as cause to effect or as

effect to cause, appears to me to place them both in the category of actualities, since a cause cannot be said to be in operation if we conceive it as a mere potentiality. I should therefore include in what used to be called the primary qualities of matter, besides spatial configuration and motion, resistant force, this phrase being preferable, in my opinion, as well as of wider application than the dubious term 'impenetrability.' My definition of the so-called primary qualities is, therefore, that they denote the adjectives or relations in terms of which *actual* physical phenomena can be described; whereas the so-called secondary qualities are properties, inasmuch as they denote *potentialities* for producing sensational effects. Thus in describing the colour of the surface of a body, we may be defining something physically actual, but we are also most certainly defining, besides, what is merely potential; viz., that if a luminous centre, such as the sun, is in such spatial relation as to radiate energy to the surface of the body in question, then assignable parts of this energy will be absorbed at the surface, and another assignable part emitted. Correlated with this physical potentiality of the body is a psychical potentiality, which must also be presented partly in spatial terms; viz., that if a living organism susceptible to light-impressions be in appropriate spatial relations to the body, there will be a visual sensation to which the name red primarily and properly applies.

The varied applications in physics of such terms as coefficient or index are obvious illustrations of what, from the logical standpoint, we regard as potentialities in contrast to actual physical process. Such terms denote what are commonly called constants; and the

common use of this term will serve to illustrate the notion of a property. Most so-called constants should more strictly be called 'variable invariables,' for only a few of them are absolutely invariable; they assume different values for different classes of bodies, although they may remain potentially invariable when applied to any one the same body. This illustrates the point that in attributing a property to a body, we imply that a certain formula can be asserted of the processes in which that body may be concerned, which formula remains unchanged on the different occasions in which the processes may take place. It further illustrates the point that bodies may be classified according to the different values of their determinable properties, as represented by the different values of the so-called constants. As an example of this, let us take the adjective 'ruminant.' So far as we predicate this adjective, we are certainly implying the existence of an organ of specific character, which could be defined anatomically or structurally in terms of spatial arrangement and, say, mechanical pressure. In addition to these elements of definition, we should further assume that the organ is persistently functioning, though the determinate mode in which it is functioning would be changed from time to time in accordance with a general formula defining the potentialities of the organ to function whenever assignable conditions may be actualised.

§ 8. So far we have been discussing the continuant chiefly from the point of view of deduction; but I propose now to treat it from the point of view of primitive induction, and to exhibit a constructive process by which, prior to anything that could be called classification of organised

or of unorganised bodies, mankind have put phenomena into groups. The symbols  $P_1, P_2, P_3 \dots$  will represent phenomena characterised by  $P$ , the suffixes being understood to indicate some kind of *order*—either temporal or spatial or both—which we shall speak of as a nexus. The first motive for grouping phenomena is the observation of some such nexus: thus when an order from  $P_1$  to  $P_2$  to  $P_3$  to  $P_4 \dots$  has been repeatedly observed to be maintained under a variety of circumstances which, in some of the different recurrences have been constant and in others have varied, then these phenomena have been grouped as manifestations of an existent agent. This observed uniformity in the temporal succession of phenomena is then inferentially extended to apply to other assumed phenomena, regarded as modes in which a single agent manifests its continued existence. The above conception of nexus therefore involves not only a preserved temporal order of phenomena, but also reference of these phenomena to a single continuant. In physical, as distinguished from psychical, phenomena there is, in addition to the temporal nexus, a spatial nexus, between phenomena presented in spatial contiguity, representing modes in which a single material body or occupant is spatially manifested. Thus when the spatial order of such characters as  $P_1, P_2, P_3 \dots$  is preserved throughout exterior processes which are changing or remaining unchanged, the group of characters is with special impressiveness taken to constitute a unity, and conceived as referable to a single occupant which maintains its form of spatial nexus, however exterior conditions may be altering.

The unity of the occupant is, however, not stably maintained with the same degree of permanence as

is attached to the temporal succession amongst the manifestations of the continuant. In this respect spatial occupants fall into different classes according to the degree in which they preserve the form of their spatial nexus: the bodies which preserve their spatial nexus in the highest degree are called solids; those that preserve it in less degree are called liquids; and those in which the spatial nexus can hardly be said to be preserved at all are called gases. These degrees of spatial nexus actually depend upon the power of bodies to sustain tension and pressure; bodies which can sustain both are called solids; liquids cannot sustain tension, but only pressure; and gases, which are also unable to sustain tension, would lose their spatial nexus altogether unless pressure, produced by external force, determined the space which they can be made to occupy. Bodies regarded as occupying spaces of defined shape and size can be divided into classes on many different *fundamenta divisionis*: for instance according as they are unorganised or organised in various different forms, or again according to their chemical composition. A body is classified according to the mode in which it maintains its unity as a whole, this unity consisting in the *permanence of mode* of presentation of its manifestations, while exterior conditions change or remain constant. When the spatial unity which leads us to conceive of a body as a single whole is dissolved by exterior force which the body is unable to resist, then each of the several parts into which the body becomes separated has its own constitution as a unit; and this constitution may or may not be generically the same as that of the whole into which the parts were previously united.

§ 9. The consideration of the different degrees of spatial nexus exhibited by different spatial occupants, leads to the notion of a single continuant-occupant being at the same time a system of *sub*-continuant or atoms. These sub-continuants constitute a system, in the sense that they preserve a certain form of spatial nexus either unchanged, or else changing in accordance with a causal formula which expresses both the immanent causality to be attributed to the several sub-continuants, and the transeunt causality to be attributed to the interactions amongst these sub-continuants. For example, the gas that is contained in a vessel is a system of sub-continuants—viz. the gaseous molecules—which severally exhibit their own immanent causality, while their spatial nexus, manifested as mutual pressure, exhibits transeunt causality. Again each molecule may be a sub-system consisting of sub-continuants—viz. atoms—which taken severally exhibit immanent causality, and taken in combination, transeunt causality. Thus, any system of sub-continuants may be regarded either as a unity or whole, whose changes are determined under the form of immanent causality; or any such whole may be conceived as a system of parts which are themselves continuants, and whose processes are to be distinguished from the transeunt processes involved in their interactions in the larger system. In future, the phrase 'single continuant' is to be understood to include a single system of sub-continuants or sub-occupants. Such a system may be ranged in an order exhibiting higher and higher forms of unity; of which the lowest form is a mechanical system, and the highest a psychical continuant. In a mere mechanism, the system as

a whole can be defined in terms of the immanent character of its parts, combined with the transeunt causality under which the parts interact; but the higher form of unity exhibited in an organism entails a much more complex interrelationship of immanent and transeunt interactions; thus each part of an organism undergoes processes which follow chemical and mechanical formulae, which remain true even when the organism is dead.

§ 10. The forms of temporal and spatial nexus observed prior to experiment to have been maintained between groups of phenomena, lead to their constructive reference to a single continuant-occupant. Primitive induction consists in the more or less tacit inference that where such forms of nexus have been found to obtain within the range of observation, the same form or degree of nexus will be maintained in all cases that have not yet been observed. This may be shortly expressed in the proposition: 'All manifestations of a given single continuant will assume a specific form of nexus'; and the assignment of a substantive-name, therefore, to any manifestation in space and time, is tantamount to a statement of uniformity. It should be specially noted that the range of uniformity here asserted does not extend beyond the manifestations of a single individual continuant. Now this uniformity is not a mere statement of the invariability by which we can infer from certain phenomena other phenomena in spatial and temporal connection with these, but it involves also a statement of causal relation between the phenomena. The causality affirmed is immanent, and is conceived as exhibiting the nature of the individual continuant itself. The phenomena in question may be said to *co-exist*, the prefix *co*



indicating that they are manifestations of a single continuant; for, as I have elsewhere pointed out, the phrase *co-existence* of properties does not mean merely temporal simultaneity in the manifestations, but includes precedence and subsequence, and also in some cases spatial relations.

§ 11. The assertion of causality does not carry with it any implication of a *uniformity* of co-existence, affirming that the causal formula which applies to the continuant under consideration would hold of other continuants. The nature of any given individual is exhibited in a uniformity of its manifestations, which again, in my view, is a causal uniformity. But this species of uniformity is not one that can be extended in application to the manifestations of other individuals, except so far as we have inductive evidence that the same form of immanent causality that expresses the nature of one individual expresses also the nature of other individuals. The *causal uniformity* which obtains between the manifestations of a single individual continuant is established by more or less direct and simple problematic methods; whereas in order to establish a *uniformity of co-existence*, the criteria of probability employed are much more exact and logically elaborate. What I have now to maintain is that there are many conclusions reached by the first method which can be said to have been established with a very high degree of probability, amounting in some cases to practical certitude. They assert persistence in the form under which the manifestations of a *single* continuant may be subsumed. Induction, on the other hand, which attributes the same form of property to one and to another continuant, cannot attain a high

degree of certitude by simple problematic methods; for what enables us to construct a class of continuants so defined, is the employment of more or less *scientific* method.

§ 12. We will now pass to the consideration of the relation of the continuant or supreme substantive to its occurrents or modes of manifestation. These modes of manifestation have variously defined characters and are connected with one another by *temporal* and spatial relations. Just as the supreme substantive is called a continuant on the ground that it continues to be manifested throughout an indefinitely prolonged period of time, within which any one manifestation or occurrent is referred to a specific sub-period; so it may also be called an occupant, on the ground that its manifestations occupy an indefinitely extended region of space, within which any one manifestation is referred to a specific sub-region. The determinate sub-period and sub-region together define the relation of one occurrent to others; whereas the time and space, of which these sub-periods and sub-regions are respectively parts, may be predicated referentially to the continuant-occupant itself. Thus when we say that a continuant exists throughout the whole of time, or an occupant throughout the whole of space, we mean that its several different manifestations are to be referred to one or another of the sub-periods or sub-regions within this whole; and if we are concerned with an animate or psychical continuant, the whole of time or the whole of space within which it exists may be itself a sub-period or a sub-region within a larger whole; since there may be some part of time during which it is not manifested in any part of space, and

there may be some part of space in which it is not manifested in any part of time. This is one of the ways in which predications (viz. temporal and spatial), which are primarily attached to existent *manifestations* are transferred to an existent *continuant* or *occupant*. We are thus led to enquire what other adjectives can be predicated of a continuant which are derived from the characterisations of its several occurrents or manifestations. To explain this we must conceive of the unity of the continuant as exhibited in causal formulae symbolisable as follows:

$$P=f_P'X, \quad Q=f_Q'X, \quad R=f_R'X, \quad \dots$$

where each equation represents a property of the continuant. The small letters *f* indicate that the property or function is defined determinately, while the capital letters *X, P, Q, R*, indicate that this same determinate property is exhibited for all determinate values which *X, P, Q, R* may assume at any specific time or place. Now as regards physical continuants, there are many properties or functions which exhibit literally the same determinate value at any time or in any place; but, in more complex organic continuants, such properties as have been denoted by the symbol *f* change from time to time. These temporal changes in the manifested properties of things are not irregular, but follow a law (dependent upon the immanent character of the things) which constitutes a property of what may be called a higher order than the properties symbolised by *f*, which directly define the causal characters of actual manifestations. In the case of complex organic continuants, therefore, any actually manifested property symbolised by a determinate *f* comes within a determinable *F* which,

in its variations, exhibits a determinate property, say  $\psi$ , of a higher order. The formula for this higher order of property will depend upon the variable—time—as also upon the different circumstances which have operated transeuntly upon the continuant from time to time. The formula itself will, however, be a formula of immanent causality, exhibiting the nature of the continuant itself as determining the kind of effects which will be produced in it under the influence of external circumstances. It is impossible adequately to represent this notion of alterable properties in symbols, but I suggest the following scheme for illustrating the conception:

$$P=\psi_P(X, Y, Z, \dots T_X T_Y T_Z \dots),$$

which degenerates into  $P=f_P(X, Y, Z, \dots)$  when the *T*'s have some determinate values. From this formula the value of *P* is determined from the circumstances *X, Y, Z*, and the times *T<sub>X</sub>, T<sub>Y</sub>, T<sub>Z</sub>*, at which these circumstances have operated. In this formula the capitals represent variables: of these variables, those in the bracket are *independent*, while the variable *P* is *dependent* upon these independent variables, of which it is a function. The small letter  $\psi$  represents a property of the higher order, having a determinate value which is constantly manifested whatever variations the variables may assume either actually or hypothetically. By the applicative principle, we infer from such a formula the actual mode of functioning of the continuant, when the determinables are replaced by given determinate values. The specific form of the function  $\psi$ , as indicating the character of the continuant itself, illustrates immanent causality; but so far as the circumstances *X, Y, Z* are due to the

operation\* of external agents, transeunt causality is involved in occasioning the actual values that they may assume from time to time.

§ 13. From the above attempted explanation of the notion of property, it is but a short step to the concept of a continuant; for the main element in the notion of thing or continuant is the permanency of functioning that can be discerned in a series of characterised manifestations, presented in the course of time, as they may be observed in a temporally continuous, or discrete, series of acts. Thus the notion of a continuant is constructed in terms of temporal connection and causal determination, and my particular views on this subject may perhaps be best explained by comparing my account with Kant's exposition in the *Critique of Pure Reason*. Kant holds that there are certain categories, such as substance and causality, under which we objectify our sense-experiences in an order of time; whereas I prefer to treat substance and causality, not as two separate categories, but as two aspects of a single principle of construction. Again, instead of adding a third category—namely reciprocity—to substance and causality, as Kant does, I include reciprocity in my account of immanent and transeunt causality. But such apparent differences between Kant's exposition and mine are not important, and readers of Kant, by putting together various parts of his exposition, would find at least hints of all that I have said. Thus, when he says with regard to his category of substance, that the idea of change involves the idea of permanence, and when this is supplemented by his schematism of causality under the form of time, his view is seen to be in close accord with my account of the way in which

temporal causality and permanency of functioning enter in the notion of a continuant; although the postulate of permanency refers in Kant's exposition to a 'quantum' rather than to the *mode of functioning* which I attribute to the continuant. In my view this permanency in the mode of functioning is inseparable from the property—or form of causality—this form being just that to which permanence is attributed; whereas Kant appears to affirm that the substance itself exists permanently, and that a second permanence is to be attributed to one of its modes of manifestation, namely to its 'quantum.' This postulate of his was, in fact, an anticipation of the constancy of mass which is a special postulate in physics; but no similar quantitative constancy can be attributed to the higher substantive entities, such as the organism or the experient. I am inclined to attribute Kant's denial of the possibility of rationalising psychology to his rather exclusive consideration of the forms in which the principles of physics can be generalised and formulated in precise mathematical conceptions. Thus my account presents a more general conception of substantive continuance, which applies equally to the notion of a conscious experient on the one hand, and to a hypothetical physical atom on the other. The unity which I ascribe to the continuant is a causal unity of connection between its temporally or spatially separated manifestations; an observed or assumed causal formula, under which the character of these manifestations may be subsumed, is the sole ground for regarding them as manifestations of one and the same continuant. I have also attempted to render clear the difficult conception of the union of permanence with change. It is natural

to ascribe change to the modes of manifestation, and permanence to the substance to which these manifestations are referred; but this is an inadequate expression of the antithesis; for, to express the matter accurately, the only things which can be said temporally to exist are the manifestations themselves; thus our first definition of the continuant is that it is merely the sum of all the manifestations. This of course does not mean that manifestations of reality are taken indiscriminately, mentally added together, and their sum called a continuant; what is meant is that certain manifestations of reality, between which a unique kind of relation can be predicated, together constitute a genuine whole or unity, to which the name continuant may be given. This type of relation, which constitutes the unity of a single continuant, is conceived primarily as one of immanent causality, while it is transeunt causality that constitutes the ground for asserting a plurality of non-identical continuants whose manifestations can be said to belong to one universe of reality.

§ 14. All the conceptions expounded in this chapter are virtually denied by a school of philosophers to-day. In particular they regard the conception of change as fictitious, and substitute for it merely differently characterised phenomena referred to non-identical dates. Whenever there is a spatio-temporal nexus between phenomena, the locating and dating of the occurrents is such that these may be conceived as a whole. Such a whole is of the kind which we have described as extensional<sup>1</sup>, and so far as extensional wholes are admitted by the scientist, no more transcendental conception than

<sup>1</sup> Part II, Chapter VII, § 8.

that of a whole constituted by the binding relations of time and space is required; and hence the philosophers who reject the conception of a continuant are satisfied to replace it by the notion of such an extensional whole. But the stability of a spatio-temporal nexus cannot, I maintain, be explained without the conception of a continuant, which, in my view, is *a priori* in the Kantian sense, and not derived from the analysis of experimental data. Given the conception, however, it is a question of mere experience to what set of phenomena the *a priori* notion is to be applied. In attempting to avoid this conception, it appears to me that my opponents alternate between a purely physical and a supposititious perceptual account of the facts. Thus in one breath they shelve the physical continuant by supposing that the percipient is observing a continuity in the qualitative changes of the object perceived; and while in this way rejecting any physical continuant, they have recourse to a psychical continuant—namely the percipient. Here I submit that the perception by any individual of certain processes offers no explanation whatever of what in objective reality determines the stability of any given nexus. Then again, on the other hand, when it is urged that the upholders of this view are all along assuming a psychical continuant—viz. the percipient—which from their standpoint must be repudiated, they, in effect, retort that it is quite unnecessary to postulate any psychical continuant, inasmuch as the nervous system itself will take the place of the ordinary conception of an ego. Here then they only eliminate the psychical continuant by reinstating the physical continuant.

## CHAPTER VIII

### APPLICATION OF CAUSAL NOTIONS TO MIND

§ 1. THE science of psychology—so far as it is purified from all reference to the physical—uses the conception of immanent causality within the systematised unity constituted by a single individual experient. In other words, pure psychology abstracts from any transeunt causality which may be actually operative in the interactions between an experient and the material world, or between one and another experient. The different phases within the experience of an individual are conceived as related temporally and not spatially; hence the form of space under which we conceive transeunt causality in the physical universe, does not apply within the individual's experience. A modified form of transeunt causality is applicable however to the interactions amongst the distinguishable phases revealed by a fundamental analysis of conscious experience. If, for instance, we distinguished between merely sensational processes, on the one hand, and active or purposive processes on the other, we might establish some kind of uniformity which would determine the course of a sensational process so far as it was uninfluenced by active purpose. Again a process of deliberation might be known to pursue a course of its own independently of the changes occurring in the sensational process. In this way a relative and partial independence might be attributed to the sensational and deliberative processes

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respectively, each proceeding according to its own law. But, if we suppose that at certain periods the deliberative process exerts a determinative influence upon the further course of the sensations, this influence is analogous to transeunt causality, with the difference that the causality is not conceived as passing across from one to another substantive existent, but only from one to another phase of experience within the unity of a single substantive existent. Another rough illustration of a similar type may be taken from associative as distinguished from attentive processes. Thus the forms assumed in the flow of images or ideas may be supposed to depend essentially in the first instance upon the time order of past experiences, and the frequency or recency of these experiences. In so far as this is so, the course of thought has a law of its own which operates independently of purposive control. But at such periods, when the felt interest of the thinking subject modifies the course of associations, and determines the flow of images or of ideas to be other than they would have been as the result of mere association, a form of mental causality is operative which is more or less analogous to transeunt causality. Those psychologists who explicitly attribute activity to the subject may be said to use the conception of transeunt causality in an even more literal sense than that which I have so far suggested: for they hold that, apart from subjective activity, mental processes would pursue a course determined on principles quasi-mechanical, these quasi-mechanical processes, constituting a sort of non-ego. Over against these, the subject or true ego is conceived as an agent having the power, which it exercises from time to time, of controlling or



modifying the processes which apart from such activity would proceed purely mechanically. This splitting up of experience into mechanism and active control tends, in my view, to misrepresent the case, if it leads to a conception of the subject as purely abstract. The subject as active must be conceived as a determinate phase of experience, which stands from time to time in definable and alterable relations to the processes that may be said to be actively controlled. The two most fundamental of these relations are called respectively feeling and cognition; according to the mode in which experiences arouse feeling, and according to the manner or extent in which they are cognised, so is the exercise of controlling activity determined. Thus feeling and cognition operate as psychical forces which are analogous to physical forces, except that the latter involve spatial relations.

§ 2. Another quite unambiguous example of transeunt causality is the action of the psychical on the physical, and the apparently simultaneous action of the physical on the psychical. We may venture to speculate that various phases of psychical process which proceed contemporaneously with physiological (and in particular neural) processes, can be described in terms of the same number of distinct determinables as the neural processes. It is therefore theoretically possible to predicate of such phases of mentality a one-one correspondence with the neural processes, where the term "one-one correspondence" is understood as equivalent to reciprocal inferability. This means that when (if ever) psycho-physical knowledge has been adequately advanced, it will be possible from a knowledge

of the character of any neural process that may have occurred, to infer the character of certain of the contemporaneously occurring phases of mentality, and conversely. This will give a restricted validity to the conception of *neutral* psycho-physiological parallelism, the word "neutral" indicating that the theory does not prejudice the question whether the assertion of correspondence implies an assertion of causal connection or not. The conception is *restricted* in the sense that, besides the phases of mentality which correspond in their changes to the changes of neural process, there are other psychical phases to which no changes of neural process correspond. If the former be provisionally called sensational experiences, I hold that these are to be clearly distinguished from cognition and feeling, which constitute the fundamental aspects of psychical process to which no neural processes correspond, and which may be provisionally defined as variable or alterable relations or attitudes towards the sense-experiences. This may be otherwise expressed in the assertion that there is no *direct* correspondence between the phases of cognition and feeling on the one side, and changes of neural process on the other side. For, if we assume that the changes of sensation are caused by changes of neural process, it follows that, in so far as these changes of sensation determine changes in the phases of feeling and cognition, there will be *indirectly* a correspondence between the latter and the neural processes. The more rigidly we insist that the phases of cognition and of feeling are occasioned in their changes by the changes in the sense-experiences to which they attach, the more clearly shall we realise the

fundamental distinction between the *direct* correspondence of the sensations to the neural processes, and the *indirect* correspondence of these latter to the phases of cognition and feeling. If, for instance, the changes of neural process are regarded as the sufficient causal determinants of the changes in the sense-experience, and these changes of sensation as sufficiently accounting for the cognition and feeling attaching to them, then there can be no remaining modes corresponding to the phases of active cognising and feeling in which the neural processes could be described as changeable. It appears to me that all the experimental work, which endeavours to establish laws connecting feeling, for example, with sensational or physiological changes, adopts precisely this same hypothesis. In these experiments an attempt is made to formulate in general terms the sort of character which the physiological processes must have, in order to account for the accompanying feeling as being more or less pleasurable or painful. It is assumed in these experiments that all the changeable modes of *neural processes* have as their correspondents changeable modes of *sensational experiences*. There is never any hint that the physiological processes could be changed also in some further mode corresponding to the changeable phases of feeling. Similarly in the case of cognition, where by this term is meant—not merely awareness of a sensation but—the cognising it as having a certain character. Thus, when I speak of changes in the cognitive phase, I mean to refer to such changes as apprehending one sensation as red, and apprehending another as blue. In this case again, physiologists implicitly assume that all the modes

in which the neural processes can be described as changeable are exhausted in accounting for the sensations *being* red or blue. There is therefore no residual mode according to which these neural processes could be supposed to vary or change which might correspond to the psychical fact of *apprehending* the sensation as being red or as being blue. A closer examination of the nature of cognition and feeling will further confirm this view; for, while the sensational processes may be supposed to be entirely accounted for by the actual neural processes that are contemporaneously occurring, the cognition and feeling which attach to these sensational processes are partly determined by past and possibly remote experiences, and therefore cannot be wholly accounted for by the present sensational and contemporaneous neural processes.

§ 3. A serious objection might be taken to the above account, on the ground that I have omitted any reference to those more central cerebral processes which physiologists describe as underlying the active phases of cognition. In this connection I will introduce the word effort or strain; and I wish to suggest that the phenomenon of effort constitutes a link between the physio-sensational mechanism on the one hand, and the subjective control which is exerted upon this mechanism on the other hand. I maintain that the phenomenon of effort or strain has both a physiological and a sensational aspect: that is to say, it involves a process in the physiological mechanism which, like other physiological processes, entails a corresponding sensation; it is therefore legitimate to speak of an effort-sensation, and the term indicates that there are certain analogies between

this kind of modification of conscious experience and other sensations such as visual and auditory. Thus, there may be differences of intensity and even of quality in the experiences of effort, correlated with differences of intensity and of locality in the underlying processes of the neural mechanism. Again, an effort-sensation resembles other sensations in the further respect that at any moment it may be more or less determinately characterised in an attitude of cognition; and this act of characterisation may reach such a limit of indeterminateness that we should say in ordinary language that the sensation was not being cognised at all. Lastly, a character may be attributed to these sensations apart from any cognition of it by the experient; that is to say, this character is unchanged by his more or less determinate cognition of it. In all these respects, effort-sensations resemble what are ordinarily known as sensations; there are two important points, however, in which they differ. Briefly these are (1) that they are subjectively initiated; and (2) that they entail directly changes in the neural processes which indirectly produce effects *intended* by the subject—the term ‘intention’ implying foreknowledge. It is important to point out that the subject does not of course know what sort of *neural processes* are taking place, although to him is to be attributed the initiation of these processes. The sense in which his activity is guided by knowledge is expressed by his foreknowledge of the effects upon his future sensations or perceptions which will be causally determined by the particular operation that he initiates in the neural processes. Thus we attribute to subjective initiation various physical effects which, for our present

purpose, may be roughly divided into two classes, describable as inner and outer respectively. The inner physical effects are those which are *unknown*, the outer are those which are *foreknown*. As physical occurrences, the inner are causally determinative of the outer; but, in our analysis of the mental processes involved, we have to maintain, what may appear paradoxical, that it is the *foreknowledge* of the outer which causally determines the *occurrence* of the inner. One, and perhaps the most fundamental aspect of this foreknowledge is when the outer effects are imaginatively characterised by the subject in their *spatial* relations. The terms inner and outer may be taken almost literally as defining what has been going on *inside* the organism, as distinguished from what will be going on *outside* the organism. When the intention of the subject has been actualised, what he perceives is presented as (so to speak) external in its spatial relations, and it is these externally manifested effects which are foreknown or prospectively imaged.

§ 4. We have said that the operation upon the neural mechanism, which appears to involve strain or effort, is initiated by the subject. In using this phrase I am not thinking of the subject in a merely abstract sense, or as a sort of transcendental ego (to use Kant's phrase); but, on the contrary, of a process precisely definable in terms of mental phase. In fact, as I have already attempted to show, it is a mode of feeling and a form of cognition which jointly determine the specific operation upon the mechanism. Under the various modes of feeling I wish to include not only the hedonic variations of pleasure and pain, according to which experiences are felt as

more or less pleasant or unpleasant, but also the modes of conation, according to which we feel more or less strongly attracted or repelled by different experiences. Again cognition should be understood to include not only the foreknowledge of the finally intended effects, but also (as higher forms of knowledge develop) the foreknowledge of the external means which must be employed to produce these final effects; the special term purposive is used to describe voluntary action of this higher kind. From the above analysis it is clear then, that in the psychical determination of physical effects, foreknowledge is involved, and we have attributed to foreknowledge real causal efficiency. If this foreknowledge could be reduced to merely physical or physiological terms, we should have to regard mental causality as an illusion; and those psychologists who hold that the changing phases of cognition are represented by corresponding neural processes, do in effect deny any genuine validity to the conception of mental causality.

§ 5. Closer examination of subjective activity introduces the notion of attention; for I hold that the most important consideration in any account of cognition is the different degrees of determinateness with which the character of an object may be apprehended in an attitude of attention. It is implicitly maintained by some psychologists that a sensational process can only properly be said to occur when the subject is cognising the sensation; so that where there is *no* such cognition, these psychologists maintain that there is no sensation; and all that could be asserted (as corresponding to the non-occurrent sensation) would be a neural process which could be defined by the physiologist. Now, whether

this contention is sound or not, it must be agreed by all that, when the character of a sense-experience is the object of cognition, it may be cognised by the subject with very different degrees of determinateness or indeterminateness. It will also be almost unanimously agreed that the function of attention is to render the object attended to more determinately cognised; and that continued attention to one and the same object does, in effect, produce this result. Increasing determinateness of cognition might thus be marked off as the *effect* of attention. When further we say that attention involves activity, and attribute this activity to the subject itself, we are attributing the *cause* of the process to the agency of the subject. But this does not explain why a more determinate knowledge of one object rather than of another is being developed; and to account for this a defined purpose is to be attributed to the subject, the achievement of which demands this further determinate cognition as means. It would be artificially formal, however, to draw an absolute line of distinction between means and end; our attention may, for instance, be momentarily diverted to an intrusive sense-experience, in which case the more determinate characterisation of the intruder may be an end in itself of the momentarily diverted attention, and not, at the time, pursued in the achievement of any more remote end. Incidentally it may be observed that if the sense-experience still continues, when attention has ceased to be diverted to it, it may operate as a disturbing factor in the attention which reverts to its previous object.

We are now in a position to distinguish two types of subjective activity. (1) The operation of the subject

upon the *sensational* processes, and directly or indirectly upon the external physical effects presented to perception, discussed in our previous general account, may conveniently be called motor, because its actual manifestation in the physical world involves a change or maintenance of spatial position. This motor activity which produces or prevents overt physical movement is to be distinguished from (2) the activity of attention, which appears to involve only a furthering or development in the determinateness of cognition. Either of these forms of activity may entail a more or less intense effort: corresponding to the first we may speak of motor effort, and corresponding to the second attentive effort; and both motor effort and attentive effort may have a quasi-sensational as well as a neural or physiological side. In actual and overt movement we can trace the neural processes which entail corresponding sensations; but in the case of mere attention, where there is no overt movement, we have to consider what may be the nature of the physiological processes which underlie the effort-experience that seems often to accompany attentive processes. In motor activity we assumed an operation upon the neural processes underlying *sensation*; in a similar way, we may perhaps assume that the activity of inner attention entails an operation upon the neural processes underlying *imagery*. If this hypothesis is to be consistently applied, our account of "restricted parallelism" must be extended to include a strict correspondence—not only between sensational experiences and their neural accompaniments—but also between image experiences and *their* neural accompaniments. This may be done without prejudging the physiological question, irrelevant to our

present purpose, whether, underlying any sense-impression and that sense-image which is a sort of copy of it is the same neural process in a different form, or locally distinct neural processes. The suggestion I wish to put forward is that such effort as appears to be experienced in inner thinking is due to the occurrence of imagery entailed in operating upon the neural processes. Without discussing this problem, I may point out a possible confusion between what is properly to be called the *effort* involved in thinking, and the difficulty of such thinking. For a given subject, the difficulty of making further progress in thinking on a certain topic—so far from implying intensification of effort—may lead him to cease thinking further on the subject. This leaves the problem still to be examined whether, so long as there *is* a continuance of the thinking process, more or less effort is involved. Evidence of such effort may not easily be found in direct introspection, and may have to be sought indirectly in such effects as a diminution in physiological or intellectual energy; i.e. in future power of doing work.

§ 6. I will now give, more explicitly, my grounds for the hypothesis that the changes of cognitive phase have not counterparts in the changes of neural process; considering in succession the three main features characteristic of even the simplest cognitive process.

(1) We may agree that perception denotes essentially the mode of cognition in which objects are apprehended in their temporal or spatial connections or relations. We may further agree that corresponding to the relations of time which we predicate of objects cognised, there are actual relations of time subsisting between the



neural occurrences which underlie the experiences whose temporal relations are apprehended in perception. The dating and temporal measuring of these neural processes by the physiologist would exhaust what could be stated about them, when their mode or locality had been defined. There appears therefore to be no discoverable further mode of neural process which should correspond to the mental act of cognitively defining the temporal relation. What holds of time, holds of space: the spatial relations between the several areas, of which the stimulation causes sensations, may be assumed to vary with the spatial relations as apprehended in the object perceived, in accordance with some formula. And the physiologist may also correlate the locality of the stimulations of the end-organs with the modes of neurosis or the cerebral localities of the physiologically central processes. This again would exhaust his account of the physiological processes, and there would be no other mode of variation which could be correlated with the act of cognitive spatialisation.

(2) Consider next the feature of elementary cognition which is involved in the act of comparison. In the simplest case this act is a cognitive determination of a relation of difference or of agreement. We may assume that the respects in which our sense experiences agree or differ correspond to the respects in which the underlying neural processes agree or differ. But as in the case of spatial and temporal connecting, there does not appear to be any further mode in which the neural processes could vary which should correspond to the act of apprehending such relations of agreement or difference; these might or might not take place without

affecting the agreements or differences themselves. This same argument applies even if those theorists are right who maintain that the only way in which we cognise the *characters* of our experience is by cognising relations of agreement or of difference between one experience and another; for here, as in the previous case, there can be no physiological correlate for cognition in general. If we pass for a moment to the higher forms of cognition which constitute the special province of logic, such relations, for instance, as those of identity, of substantive to adjective, or of cause to effect, it is still more impossible to conceive of modes of variation of neural process which could be correlated with the occurrence of such acts of conceptual cognition.

(3) The third feature of the cognitive process to which I wish to draw special attention is the variability of the degrees of determinateness or indeterminateness with which an experience may be cognitively characterised. This feature of cognition may be said to include the features treated under the two previous heads, and to constitute the most fundamental ground for our denial of a physiological correlate of cognition in general. For the cognitive determination of temporal relations, or relations of agreement and difference, etc., is to be regarded as a case of the further determination of relations which were earlier apprehended in a comparatively indeterminate form. Now the actual experiences, and consequently the corresponding neural processes, are *in fact* determinate in character; so also are the relations amongst them. How, then, can this determinateness of character and of relation be combined with varying degrees of indeterminateness that should be correlated

with the varying degrees of indeterminateness of cognition?

§ 7. It appears to me that the reason why physiologists and psychologists never properly face the problem of the neural correlate of cognition, is because they virtually identify ideas with images. This confusion is especially apparent in discussions about so-called 'general ideas,' 'abstract ideas,' 'conception,' or 'conceptual thinking.' The older nominalists denied the possibility of such forms of thought or ideation, and maintained that the only mental content which can actually be asserted in abstract thinking is the *word* heard, or uttered, or represented in imagery. For modern psychology the problem of the relation between language and thought is still a burning question. So far as language is concerned, the simplest case to take for illustration would be that in which we characterise a sense-experience, say as being red, and, in this act of characterisation, utter or image the *word* red. This process is partly explained by the formula of association, and the associative process may be safely assumed to have a neural correlate; so that the mental association between the sensation red and the word red may be correlated with some connective process, taking place between the central or cerebral processes which underlie the visual sensation and the word-imagery respectively. But even in this simplest case, it must be pointed out, that to define the mental process as merely an association between the word and the sensation, is wholly inadequate. On the occurrence of the sensation, it is not only the image of the word which is aroused in the mind of the thinker; but he mentally connects the word with the

sensation in a form which could be expressed in some such proposition as: 'The word red stands for the quality characterising this sensation.' In fact, passing from this simplest case to the higher forms of thought which may be accompanied by language, the verbal expression of a proposition may be taken to represent the universal *form* of an act of thinking. Association and its probable neural correlate are involved in so far as the words, comprising the entire sentence, can be said to be associated with one another and with the objects to which they refer, so as to constitute a whole. But, if we examine the mental process, we find that the sentence is not merely a whole for the thinker, but a *significant* whole. Mere association might give an adequate account of a combination of words which was mere nonsense; it cannot account for the added psychical fact that the sentence is understood as having meaning. Now an understanding of the meaning of language, and in particular of the sentence as denoting a proposition, is what is meant by thought or ideation. If the physio-psychical process of association is adequate to explain how the words of the sentence come successively before the mind, what can be the physiological process correlated with the act of understanding its significance?

§ 8. The distinction between physio-psychical processes and those which I attribute to subjective initiation is best illustrated by those reflex processes which entail various forms of consciousness, for here the contrast is sharp and unmistakable. Let us take for example the condition which we speak of as a tendency to sneeze; this is a physiological condition which is certainly accompanied by several definable phases of consciousness:

(1) There is in the first place a characteristic sensation which is distinguishable for instance from that accompanying the physiological tendency to cough.

(2) Like other sensations, at the time of its experience its character may be cognitively defined both as regards quality and locality by the experient; and it will be so defined if any interest or purpose prompts him to direct his attention to it.

(3) So long as the physiological tendency to sneeze remains a tendency, there is an element of feeling which could be called discomfort.

(4) There is normally anticipatory imagery or precognition of what will almost immediately occur sensationally when the sneeze actually takes place. This foreknowledge of the prospective sensation involves auditory and other forms of imagery.

We may now pass from what is a direct analysis of the modes of consciousness accompanying the physiological tendency, to a consideration of the causal processes which may follow. In the absence of any inhibitory act on the part of the experient, the sneeze will take place; and in this case the causality involved is purely physical or physiological. But the sneeze has a *sensational* as well as a physiological aspect; and to this aspect the term *conative* has been mistakenly applied; the term conation being taken as equivalent to felt tendency. But this definition appears to me to involve confusion; for the phrase 'felt tendency' has been used to describe the form of conscious experience which accompanies a process which will normally terminate in an explosion; whereas it is generally agreed that *conation* is a form of consciousness of which the causality is

psychical and not merely physiological. In the case of the sneeze, the word 'tendency' alone would express a merely physiological fact; and when conjoined with the word felt, it can signify only a precognition of the subsequent mode into which the sensational process accompanying the physiological fact will develop. In the physiological case, discomfort and foreknowledge play no part as *causal* factors in determining the further process; psychical causality would only enter therefore in an effort to inhibit, repress, retard, or moderate the sneeze. The exercise of this effort is conditioned by precisely the same factors as operate in any case of voluntary effort; namely by desire for a more or less specifically cognised end, and a knowledge of the immediate means. In the case before us, the conative process proper operates in resisting or opposing a purely physical or physiological tendency: it might be compared with the effort of a man to open an umbrella in a strong wind. My purpose in the foregoing analysis is to distinguish this type of case, where conation operates in opposition to a merely physiological tendency, from that in which one conative tendency conflicts with another. These two cases are apt to be fatally confused; for they agree superficially not only in the point that one tendency operates in opposition to another, but also in the further point that the struggle between the operations of the two tendencies is normally accompanied by an additional psychical factor of feeling—either mere discomfort, or a highly intense feeling amounting to pain: e.g. any moderate degree of discomfort which may precede the actual sneeze is considerably heightened in the process of endeavouring to suppress it.

A case which resembles more closely still the voluntary attempt to inhibit a reflex process is the stimulation of an emotion; but here the parts played by the physical and psychical factors are reversed. Thus when cognition of the circumstances has aroused conation, and this has naturally developed into specifically directed purpose, there occurs what is termed an emotional experience if this directed purpose is accompanied by irrelevant or perturbing organic processes, which may be assumed to be partially reflex. Thus, in the case of emotion, the physiological causality is apt to interfere with the psychical causality manifested in the form of purpose; whereas in the case of the sneeze it is the purpose which interferes with the reflex process.

§ 9. We may now turn our attention to the highly complex processes of conative conflict, and for the purpose of this analysis I shall introduce a simple mode of symbolism. It may be taken to be the general case that when there are two alternatives either of which can be actualised, there will be in each alternative some aspect or circumstance which is attractive, and some other aspect or circumstance which is repellent. If the aspects felt as attractive be represented by the symbols  $a$  and  $b$ , and those felt as repellent by the symbols  $a'$  and  $b'$ , then  $ab'$  will stand for one alternative, and  $a'b$  for the other; and the conflict will be symbolised:

$ab'$  versus  $a'b$ .

It is of psychological importance to regard the two symbols  $a$  and  $a'$  as positively opposed modes of the same generic or determinable aspect; because the mere negative *non-a* could not be represented in thought or imagery with a felt repulsion or attraction; what arouses conation

must have some positive content. Only in the limiting case, therefore, where the felt conation was indifferent and could be measured as zero, could we apply the symbol  $a'$  or  $b'$  to denote merely *non-a* or *non-b*. In order to measure the felt conation we will use the Greek letters corresponding to the Latin, with the sign (+) to stand for the aspect felt as attractive, and the sign (−) for the aspect felt as repellent. Then, if the symbol  $c$  stands for the felt intensity of the conation,

$$\begin{array}{ll} c \text{ of } a = +\alpha; & c \text{ of } a' = -\alpha'; \\ c \text{ of } b = +\beta; & c \text{ of } b' = -\beta'. \end{array}$$

The resultant conation of each of the alternatives is determined jointly by its attractive and repellent aspects: it may be shortly symbolised as  $rc$ . We thus reach the results:

$$rc \text{ of } a \text{ with } b' = \alpha - \beta'; \quad rc \text{ of } b \text{ with } a' = \beta - \alpha'.$$

Hence:

$$\begin{array}{l} \text{acc. as } \alpha - \beta' > \text{ or } < \beta - \alpha', \\ \text{i.e. acc. as } \alpha + \alpha' > \text{ or } < \beta + \beta', \\ \text{so is } rc \text{ of } ab' > \text{ or } < rc \text{ of } a'b. \end{array}$$

We shall of course assume that the stronger resultant conation is that which wins in the conflict. Now there are occasions when the subject is indifferent about the issue of two conflicting conations. For example, if I am in doubt which to choose of two possible entertainments, say a concert or a game of bridge, then I may be deliberating merely in the sense that I allow the attractive and repellent aspects of the two alternatives, as represented in imagination, to work themselves out, without reference to consequences or other considerations. Deliberative process of this kind, in which the inner working of conative tendencies is passively watched, may be called non-moral deliberation. When, on the

other hand, the subject is not indifferent as to which of the conations will prevail, he would seem to have the power to decide the issue. The desire, or conative tendency which may be said to urge him to exert this power, and thus to modify the strength of his primary desires or conations, may be called a *secondary* (or perhaps moral) desire or conation; and we will now examine the practical means by which this secondary process is exercised. On the supposition that it is the first-mentioned alternative that is to be enforced, it will be seen that what is required is to strengthen the felt attraction to *a* and the felt repulsion to *a'*, and at the same time to weaken the felt attraction to *b* and the felt repulsion to *b'*. Thus it is not only the attraction to *a*, but also the repulsion to its alternative *a'* which must be felt more strongly, if the will is to accept *a* and reject *a'*; while if *b* is to be rejected in spite of its attractiveness, and *b'* to be accepted notwithstanding its objectionableness, then the felt force of these factors must be weakened. I attach the highest importance to this double reference to the repulsive as well as to the attractive tendencies; and on this point it appears to me that James—whose symbolic representation of the process agrees to some extent with mine—wrongly represents the matter. I agree with him that an act of attention, which renders more vivid the imagery or more determinate the idea connected with the various aspects of the two alternatives, is the essential factor in the process. But in my opinion, James misrepresents the case when he assumes that the single effect of rendering to oneself an aspect more vividly is to strengthen the inclination to actualise that aspect. This can surely

only be the case when the aspect strengthened by attention is itself attractive; for if it is repulsive, then the effect of attention is to strengthen the felt repulsion. In point of fact, in many cases of moral conflict, it is the more determinate thought of what will in some sense pain us in the alternative which our moral attitude directs us to reject, that helps us to decide; rather than the more determinate thought of what will please us in the alternative which the moral attitude prompts us to accept. In the general case we shall include both factors.

Now this power of modifying primary desires and aversions by direct attention is one of the most conspicuous forms in which the ordinary man claims freedom of the will. Since the mere uninfluenced force of his desires and aversions does not inevitably determine the issue and, by exerting the power he possesses over the direction of his attention, he is able to influence the ultimate decision, he holds that therefore there is free will. This very important sense in which freedom of the will has to be maintained does not infringe the principle of causal determination which we attribute to volitional processes: it is not a mere accident or matter of chance whether the secondary desire does arise with a strength sufficient to change the issue; the agent's secondary desires will have had a set of antecedent causal conditions similar to those which we ascribe to the primary desires. Tracing this causal chain backwards, we have of course to presume potential conative tendencies present at the early periods of the developing experience; and science is here, as in other branches of psychology, supremely ignorant, the actual causal conditions constituting a problem for investigation.



§ 10. In this analysis I have referred exclusively to the conative aspect of deliberative volitional processes; but logically speaking, it would have been perhaps more correct to define the cognitive processes before analysing the conative; for the judgment or knowledge which is presupposed when we speak of the alternatives within the range of possible actualisation by the agent, is totally independent of the conative aspects. It is only after judgment with respect to the possible ends has been so to speak impartially exercised, that the conative forces moving in one or the other direction, begin to operate. While, therefore, the cognitive processes without the conative would be inadequate as determinants of the will, just as would the conative without the cognitive, it is nevertheless true to say that the forms assumed, and the prior conditions which account for the forms assumed by the judgment, are totally and absolutely independent of the conative processes. Psychologists who have given a clear enough analysis of cognitive deliberation have made, what appears to me to be, the fatal mistake of attempting to reduce conative deliberation to the same type of formula. The cognitive aspect of a deliberative process is concerned merely with the known or accepted causal conditions for actualising any supposed end, and this process of judgment has the same conative impartiality as any scientific problem—theoretic or practical. That this cognitive or intellectual process is to be assumed, seems to me to raise no controversy. I have therefore laid emphasis upon the conative aspect of deliberation since I hold that it is to this process that causality essentially applies. In this view I am opposed

to those psychologists who maintain that the will is free in the sense that it can act on mere judgment without any conative urging; and it is upon this issue that the burning controversial problem essentially depends.

§ 11. The above general reference to judgment as essential in the higher volitional processes must be supplemented by a consideration of the different kinds of predicates and their correlated subjects which together constitute the various types of propositional content. So far the judgments entering into the process of deliberation to which we have alluded have been those directed to physical or at any rate external conditions, predicating of these, characters quite independent of mental reference. But the judgment which distinguishes the higher human volitions attributes *value* to possible existents, and in certain relevant cases comparative values to different alternatives; such judgments predicating of their objects characters which are intrinsic to them, in the sense that they are entirely independent of the likes and dislikes of the person judging. Without entering into controvertible issues, it will be universally admitted that when objects are characterised by such adjectives as good or beautiful, they can properly be said to be raised into a realm of reality removed from that realm in which reference is made merely to predicates based upon qualities of sensation, or upon the scientifically developed properties of continuants. At any rate these adjectives 'good' and 'beautiful' are imposed upon their objects in an act which is quite other than the analytico-descriptive characterisations made by what we may call science; and apart altogether from any influence upon volition, this species of judgment has unique features,

which distinguish it from the type of judgment with which the simpler logic is mainly concerned. With regard to the influence of such judgments upon conation, it may be that an attitude is necessarily evoked which tends to stimulate the thinker to produce so far as possible the kind of object to which value is attached in his judgment. If so, a judgment of value of this kind may be said to be by itself the sufficient cause of a direct act of will. Where, on the other hand, the judgment is not accompanied by a felt urgency sufficiently strong to overcome conflicting tendencies, it may still be a pure judgment of value. What, in my opinion, constitutes the importance of judgments of this kind, is that where any causal relation between the judgment and the conation subsists, it is the character predicated which causes the conation, and not conversely, the conation or felt tendency to actualise the object, which causes the judgment of value.

## CHAPTER IX

### TRANSEUNT AND IMMANENT CAUSALITY

§ 1. To understand the distinction between transeunt and immanent causality it is necessary to have grasped the conception of the continuant; and to illustrate how the continuant functions in this connection, it will be convenient once again to analyse what is meant in physical science by movement. We may speak for instance of points *A* and *B* being occupied at one instant of time, and the points *C* and *D* unoccupied; while at a subsequent instant, points *C* and *D* are occupied. In the temporal interval from one instant to another something physical has happened to which the name movement is given. But such movement cannot be unequivocally described unless we are able to distinguish between two such cases as first a movement from *A* to *C* and from *B* to *D*, and second a movement from *A* to *D* and from *B* to *C*. Unless we know which of these alternatives is the correct description, our conception of what has happened in the time-interval is undefined, and no subsequent events can be inferred without presuming one or other of these alternatives; so that, in constituting the event called movement, we must assume *something* that moves, to which the name particle may here be given. The one alternative then is that a particle has moved from *A* to *C* while another particle has moved from *B* to *D*; and the opposite alternative that a particle has moved from

*A* to *D* while another particle has moved from *B* to *C*. And when we speak of two particles we conceive of each as continuing to exist and as retaining its identity with itself and its diversity from the other; so that terms such as this, or that, or it, involving reference to the same particle, are required to describe what we hold to be the character of the event.

§ 2. From this elementary illustration of a continuant, we may pass directly to an illustration of immanent causality. Thus, when the movement of a particle from *A* to *B* during an interval of time is followed by a movement of the same particle from *B* to *Y*, the law or formula in accordance with which the nature of the former movement determines that of the latter exhibits immanent causality; i.e. the causality in which the cause occurrence and the effect occurrence are attributed to the same continuant. The law in this case is known as the first law of motion, and it can be briefly expressed thus: the speed and direction of the movement of a particle is maintained unchanged from one period of time to another. The empirical establishment of this formula presupposes that no other form of causality intervenes. But when, in the physical domain, one particle is regarded as causal agent and the other as patient, in the sense that the movement of the latter is conceived as the effect occurrence, while the position of the former relatively to it constitutes the cause occurrence, a different notion of causality is introduced, and this we shall call transeunt. Here then the cause occurrence and the effect occurrence are referred to different continuants, whereas in immanent causality cause occurrence and effect occurrence are attributed to the same continuant.

This illustration serves further to indicate what may be assumed to be universally applicable, that any concretely described causal process must be analysed into a conjunction of transeunt and immanent causality; and neither types of causality are to be found actually separate.

§ 3. We now pass to a somewhat complicated physical illustration; namely the case of a gas of which the pressure, volume and temperature are conceived as its alterable states. The gaseous body to which we here refer is not a simple or ultimate physical continuant such as a particle, but consists of an indefinite number of ultimate constituents to each of which the name molecule is familiarly attached. The spatial relations amongst these molecular constituents are alterable, so that the gaseous body as a whole may be said to have an inside, and the terms pressure, volume and temperature are therefore permissible as defining its alterable states. Before we proceed to consider some actual process of experimentation upon such a gaseous body, it may be pointed out that experiment itself implies transeunt causality; for the experimenter employs physical agents whose movements he himself controls, and these produce in the material operated upon, effects which would not have been produced apart from this external manipulation. If, in the simplest case of the laboratory, the bodies there occupying space—being conceived in their combination as a whole—were left to themselves, then the changes which would take place would come under the head of immanent causality. But when the experimenter, by manipulating other bodies which he can control, produces effects which modify the

course of immanent causality, these must be described as transeunt.

Let us first consider the case where the pressure upon the gas remains constant, and the experimenter, by means of controlling instruments, alters the temperature and awaits the effect, exhibited as a change of volume, in the gaseous body. Here the mathematician briefly expresses the formula of causality in the equation  $p v = k t$ , where  $p$ ,  $v$ , and  $t$  represent respectively the pressure, volume and temperature of the gas, and  $k$  is a constant coefficient which measures a certain unchanged property specific to the gas under experimentation<sup>1</sup>. This familiar mathematical formula is inadequate, however, to express the joint transeunt and immanent causality which we propose to investigate. I shall therefore symbolise two pressures, temperatures and volumes, one applying to the gaseous body itself—which I shall call internal—and the other, which I shall call external, to the surrounding body or envelope, the suffixes  $i$  and  $e$  being employed thus  $p_i$ ,  $p_e$ ,  $v_i$ ,  $v_e$ ,  $t_i$ ,  $t_e$ , to represent one and the other continuant. The case before us is that in which gas is contained in a cylinder, the volume of which is free to change as the weight placed on the top changes. The temperature of the gas  $t_i$  is then determined by the temperature  $t_e$  of the cylinder, which may therefore be regarded as agent in the process described as the conduction of heat which causes  $t_i$  to equal  $t_e$ . Here, then,  $t_e$  being referred to the surrounding body as cause, and  $t_i$  to the gaseous body as effect, we have transeunt causality. Similarly the weight of the piston, of which the experimenter has direct control, represented by the pressure  $p_e$ , determines the pressure  $p_i$  in accordance

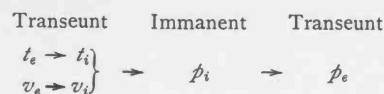
<sup>1</sup> See Part II, Chapter V, § 9.

with the process described as the transference of pressure which causes  $p_i$  to equal  $p_e$ . Here, then, there are two separately conjoined transeunt causal factors  $p_e$  and  $t_e$ , each of which entails as a separate effect  $p_i$  and  $t_i$ . Next the conjunction of the two factors  $p_i$  and  $t_i$  characterising the gas itself, causally determines the effect  $v_i$ , in accordance with the nature of the gas, thus exhibiting immanent causality. Lastly  $v_i$ , which is the volume of the gas, determines as its effect the volume occupied by the cylinder which will be  $v_e$ , since the expansive property of gas causes  $v_e$  to equal  $v_i$ . In other words, this last causal process is again transeunt, but it is from the gas as agent to the cylindrical envelope as patient. The whole process, then, may be schematised as follows:

$$\begin{array}{ccccc}
 \text{Transeunt} & & \text{Immanent} & & \text{Transeunt} \\
 \left. \begin{array}{l} t_e \rightarrow t_i \\ p_e \rightarrow p_i \end{array} \right\} & \rightarrow & v_i & \rightarrow & v_e \\
 \text{under the formula (say) } v_i = \frac{k t_i}{p_i} \text{ (immanent),} & & & & \\
 \text{and } t_i = t_e; p_i = p_e; v_e = v_i \text{ (transeunt).} & & & & 
 \end{array}$$

§ 4. A new problem is at once suggested by the above immanent formula, which connects together the alterable volume, temperature and pressure of the gas, showing them to be related independently of any action upon the gas from without. In Part II under the heading of functional deduction, the notion of the reversibility of cause and effect was treated in its mathematical or deductive aspect. This same principle is illustrated in the case before us by taking indifferently  $t$  and  $p$  as independent of one another, and  $v$  as dependent jointly upon them; or  $t$  and  $v$  as independent of one another, and  $p$  as dependent jointly upon them; and we have now to reconsider the principle under its inductive and causal aspect. In the above account of the experiment, the

volume of the gas is represented as effect, and its temperature and pressure as the two cause-factors; but the problem arises, since the factors  $p$  and  $v$  are manifested simultaneously, how to determine which of them is to be called cause and which effect; and indeed philosophical criticism of the conception of causality frequently suggests the view that where the cause occurrence and the effect occurrence are represented as simultaneous, there is no principle for deciding which of the two occurrences constitutes cause and which effect. Now the general principle whereby I distinguish the cause from the effect where manifestations are temporally coincident, is developed from the distinction and connection between immanent and transeunt causality. To explain this point, let us turn from the above described experiment of a cylinder which is allowed to increase in volume by a movable piston, to one in which the experimenter encloses the gas in an inextensible envelope. The two experiments will agree in respect of the temperature process, i.e. in either case the surroundings are at a certain temperature  $t_e$  which, through conduction, will produce a temperature  $t_i$  of the gas equal to  $t_e$ . But whereas in the previous experiment, the change in volume  $v_i$ , produced by the increased temperature, causally determined  $v_e$ , in the new experiment, where the volume of the receptacle is controlled by the experimenter,  $v_e$  causally determines  $v_i$ . The whole process in this case may be schematised just as in the other, by merely interchanging  $p$  and  $v$ , as follows:



Now in comparing the two schematisations from the point of view of the causal process to be explained, it may be asked why, while  $t_i$  constitutes a cause-factor in both cases, yet in the first case  $p_i$  is said to function as the other cause-factor and  $v_i$  as the effect, in the second case  $v_i$  is said to function as the other cause-factor and  $p_i$  as the effect. Why is it that in one case  $p_i$  to  $v_i$  stands as effect to cause, and in the other case as cause to effect? In the first experiment  $v_i$  was not the cause of  $p_i$ , because  $p_e$  was the cause of  $p_i$ ; in the second experiment  $p_i$  was not the cause of  $v_i$ , because  $v_e$  was the cause of  $v_i$ . So long as we are only concerned with the alterable states of the one continuant—i.e. so long as we are concerned only with immanent causality—there is absolutely nothing to determine which among the co-variable states is to be called cause and which effect. But as soon as we refer to the surrounding influences, and thereby take into consideration transeunt causality, then that state of the gas which is the immediate effect of the state of the surroundings stands as cause relatively to the other inner states of the gas. Briefly we may repeat the schemes omitting the points in which they agree: in the first experiment  $p_e$  causes  $p_i$ , and therefore it is  $p_i$  that causes  $v_i$ ; in the second experiment  $v_e$  causes  $v_i$ , and therefore it is  $v_i$  that causes  $p_i$ . This is generalised in the following principle: when the co-variable states of a body are causally determined in accordance with an immanent formula, then those variables which are separately effect-factors in the transeunt process must constitute the cause-factors jointly in the immanent process.

It is necessary here to point out that a totally different



account of the transeunt and immanent processes which the gas undergoes is required when (in accordance with the kinetic theory) we conceive temperature as molecular kinetic energy, etc. For, here, the free linear movement of each molecule illustrates the process *immanent* to that molecule; and, at the instant of its collision with another molecule, is illustrated an inter-molecular *transeunt* process. Moreover, in this analysis, the events are conceived as successive and not simultaneous. This more ultimate account, however, in no way impairs the validity of the above.

§ 5. It may be helpful to pass from this purely physical illustration to a case of psycho-physical causality which has been much discussed in recent times; viz. as to the relation of an emotion to its so-called expression. Before the James-Lange theory was propounded, emotion was conceived merely as a mode of feeling determined essentially by the cognition of a situation as such or such. This analysis disregarded the perturbing concomitants of such experiences as those of fear and anger; thus while in two such contrasted experiences, the nature of the situation was held to be an object of differentiated cognition on the part of the experient—a differentiation which accounted for the corresponding difference in the purposive acts which ensued—yet in the analysis of the two experiences, no place was given to perturbing organic processes, which were regarded as a mere bye-product of the emotional state. Now James held, in my opinion correctly, that the apprehension of a situation of danger which leads merely to adaptive purposive action, does not constitute a state of fear; nor would a situation in which an experient judged himself to have

been injured or insulted constitute a feeling of anger, if it simply led to correspondingly purposive action. A mere cool intellectual judgment—which is not altogether a human impossibility—could not properly be called an emotional state; in order to constitute an emotion, the presence of perturbing organic processes which produce corresponding organic sensations is an essential—or even in James' view apparently *the* essential—condition for the emotional consciousness. In my opinion the only error in James' view is that, while it is true that the cognition of the situation without accompanying organic disturbances would not constitute an emotional state, yet neither would the organic processes constitute an emotion apart from some corresponding form of cognition of the situation. Emotion requires the presence of both cognitive and sensational factors.

Agreeing so far with James' account of the concomitant factors concerned in an emotional experience, we proceed to consider how the various elements are causally related. If it be admitted that the immediate initiative of an emotional experience is the cognition of a situation as being of a certain kind, then this cognition partially illustrates transeunt causality, inasmuch as something presented to perception or imagination occasions the content and form of the cognition. The cognition to which a separate cause has thus been assigned, must be assumed to arouse a conative process, for on no other hypothesis, I think, can emotion be explained; and this conative process, being a manifestation of the nature of the experient, illustrates immanent causality. At this point we again assume a transeunt psycho-physical

process, but in the reverse order—viz. from the conative tendency as cause to the physiological process as effect. Thus the conative tendency being aroused suddenly or with intensity, the immediate consequent physiological effects are vaguely, widely, and intensively diffused, in such wise that the physiological disturbances accompanying vastly different kinds of cognised situations have many factors in common. These emotive effects are to be sharply contrasted, in my view, with the purposive effects arising out of the thought element involved in the conation; although this purpose, like the emotive disturbances, constitutes a psychical cause of a physical effect. In this analysis, the conation as a psychical process is seen to be the effect of an external cause, and in its turn, the cause of whatever specific occurrence may thereafter take place. We shall not say, therefore, after the old fashion, either that the emotion of fear causes the physiological disturbances, or that the physiological disturbances cause the emotion; for the emotion is not simple, but a compound of cognitive, conative and sensational factors.

§ 6. Having illustrated how the notions of transeunt and immanent causality are employed in physics, we will now consider the much more complicated case of psychology, assuming the philosophical position known as dualism, which regards the psychical continuant as something with a nature fundamentally distinct from that of a physical continuant. Just as the one indubitable illustration of a physical continuant is the particle, so I shall assume that the only indisputable psychical continuant is an experient, or (what for the present I wish to take as synonymous) a person, a mind, a self, or an ego. Now the

person and the particle agree in being what I call a continuant, namely something which continues to exist and to stand in relation to what changes in the course of time. The continuant, in either case, must be distinguished from its property, since the property may either remain unchanged, or may change within a given period of time; the terms unchanging and changing, therefore, apply to the properties (as well as to states and relations), and not to the continuant itself. Of the particle, physicists maintain that it has one property, namely mass, which is unchanged; and the abstract dynamic theory, which has held the field almost uninterruptedly \*since Newton, is that mass, besides being the only unchanged property of a particle, is the *only* property that can be attributed to a particle as such. Whether ultimately all particles have the same mass; or whether different particles have different masses, is irrelevant to our present purpose; and I shall assume provisionally that in either case mass is a property of the particle, which is to be distinguished from any complex quantity definable in terms of motion and in particular of acceleration\*.

Contrasting the notion of a particle with that of the psychical continuant or person, we may say of properties attributable to any given person in the first place, that they will not agree determinately with the properties of any other person; and that in the second place, they are subject to change in the course of time. Thus the concept of a psychical continuant differs from the outset in two important respects from that of the physical

\* The modifications of Newtonian physics at the present day strengthen—rather than weaken—the form of my logical analysis: the instructed reader may easily make the requisite corrections.

continuant, in that (1) no property of a psychical continuant is determinately the same as that of another; and (2) that no determinate property attributed to a psychical continuant remains unchanged during an indefinite period of time.

But a far more important and far reaching distinction between the two, is that a psychical continuant may be said, metaphorically speaking, to have an *inside*, while the physical continuant has not. In other words, apart from the property of mass, nothing can be predicated at any time of a particle except its position relatively to other particles, and the change in its relative position as time lapses; \* while we may speak properly of the complex analysable *states* of the psychical continuant. Thus the term change, when applied to a particle, refers solely to the external relation, position\*; but when applied to the psychical continuant it refers predominantly to alteration of state. It is true that science speaks of the state of a material body, where the body is conceived as containing a number of particles; but when the term 'temperature,' for example, is used to denote such an alterable state of a body, it means nothing more than the mode of agitation of the particles which constitute the body. Since this mode of agitation (i.e. of relative movement of the particles within what in many cases continues to be the same body) is subject to change, the conception of an alterable state of a material body is legitimately applicable to its temperature. Other conceptions have been introduced into physical science, resembling temperature in these respects; i.e.

\* This again requires modification in the light of present-day physical science.

they represent alterable states of a body, logically distinguishable both from the aggregate of particles which constitute the body itself, and from the properties which can be attributed to the body as distinct from the mass attributed to each of the particles of which it is composed.

The property of a composite body—i.e. a body composed of an aggregate of particles—is to be defined, not as anything actually manifested in time and space, but as a rule in accordance with which the changes actually manifested in time and space can be formulated. Material bodies may be grouped according as they continue to be composed of the same particles—such bodies being called inorganic—or according as they are composed of minor parts differing from time to time, these being assimilated from external bodies, and uniting in modes regulated by a *property* of the body, i.e. a rule of behaviour formulating the processes of these minor parts. Such bodies are called organic.

§ 7. The further consideration of the character of psychical causality leads to the introduction of at least two fundamental factors *uniquely* characteristic of mind, although a tendency has prevailed throughout the history of philosophy, to import into interphysical relations these conceptions which are true with certainty and immediateness only of psychical processes. In the first place to mind or consciousness or experience, I attribute efficient causality—a notion which can have but dubious application in the physical sphere, while there can be no doubt that something equivalent to power is a mental factor which can and does influence the course of physical events. And secondly, there are

(the processes of cognition, judging or thinking, which are of still more importance both as causally determinative, and because, regarded as effects manifested as occurrences in time, they come under a special kind of ontological determinism. Volition, which is a psychical act, determined jointly by conative and cognitive processes, exhibits both these peculiar characteristics of mind, and it is directly determinative of physical occurrences. From the time of Aristotle, the second unique aspect of mental process mentioned above has been described as final cause; and here, as elsewhere, I hold that a strictly factual account, and not a merely metaphysical explanation, is to be given of this notion. In final causality, the idea of *E*, an effect or end, is an essential causal factor in the actualisation of *E*; but at the same time, the whole significance of the conception of final cause is that it is in itself an efficient cause. When an occurrence is explained in terms of the end which a conscious and thinking being has in view, the end or final cause would appear to be functioning as efficient cause; but it must be borne in mind that the mere idea of an end can only constitute an efficient cause of the actualisation of that end in so far as it involves an act of will which, in my analysis, constitutes a crucial occurrence within the psychical processes which take place in the course of time.

§ 8. Up to this point I have sketched the logical conceptions employed in a general philosophical account of psychical processes, and have therefore only raised problems open to philosophical as distinct from strictly psychological controversy. A specifically psychological explanation of the way in which feeling and desire enter

along with cognition or thought as jointly determining the critical act called volition, is given elsewhere in this volume; and at this point I have merely put forward a brief analysis of an act of volition as guided by cognition and motivated by conation, as the most important illustration of purely immanent causality.

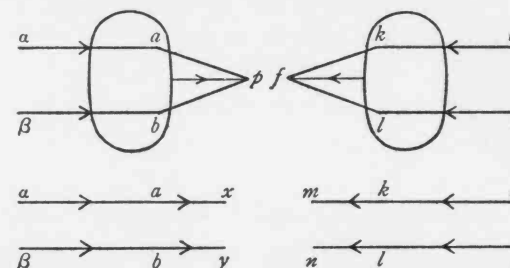
In further explication of the notions of transeunt and immanent causality in science, it must be pointed out that when the action of one continuant *x* upon another continuant *y* exhibits transeunt causality, the mode in which the states of *y* are thus causally determined cannot be regarded as dependent *merely* upon its relation to *x*, since conjointly with this relation, the mode depends upon some character peculiar to *y* itself. If this specific character of *y* be called immanent, we have an illustration of the way in which along with transeunt causality an immanent factor enters. But, when we conceive of a plurality of continuants as in some way constituting a single continuant, then, although certain causal processes are correctly conceived as immanent relatively to this unitary whole, yet they must be conceived as transeunt relatively to the several constituents. The transeunt interactions between the several constituents may or may not be known at certain stages of scientific development, and when unknown we are limited to conceiving the processes as immanent within the composite whole, while the further theoretically possible knowledge which would resolve the immanent causality into transeunt processes amongst the constituents, would not upset our previous application of immanency, but would represent a more penetrating and ultimate knowledge of the facts. An example of this alternate way

of representing causal processes has already been given. Perhaps the principle called the conservation of energy would illustrate the matter as well as any other. This principle conceives of a system of particles or bodies acting on one another, sometimes in highly complicated modes, preserving throughout all its changes a constant quantum called energy; and thus the formula of conservation of energy represents immanent causality, inasmuch as it regards the system as a whole, and as not transeuntly operated upon from without. At the same time, detailed knowledge of the forms of energy which *change* in the causal process, may be conceived under transeunt causality as operating amongst the particles of the system. This illustration from physics may be compared with the economic conception of a society of persons. Thus the formula according to which prices of commodities are maintained unchanged so far as the community is not transeuntly operated upon by other communities, is analogous to the conservation of energy; but the further analysis of the processes of exchange and contract between person and person presents the facts more ultimately and more exactly as involving transeunt relations between the persons; just as in physics processes immanent to the whole are more profoundly defined in terms of transeunt causality as regards its parts.

## CHAPTER X

## CONVERGENT AND DIVERGENT CAUSALITY

§ 1. THE whole topic of causal determination may be approached from a different point of view by considering the complex relations of interdependence amongst factors of events such as the terms cause and effect are familiarly used to describe. It will simplify the exposition of this aspect of the problem to introduce a little elementary symbolism, and throughout this chapter the reader is asked to bear in mind the following diagrams:



If both a change in *a* alone and a change in *b* alone would entail or point to a change in *p*, where *a* and *b* are cause-characters and *p* an effect-character, then we shall speak of the *convergence* of the cause-characters *a* and *b* towards *p*. In the same way, if both a change in *k* alone and a change in *l* alone would entail or point to a change in *f*, where *k* and *l* are considered as effect-characters and *f* a cause-character, then we shall speak of the *divergence* of the effect-characters *k* and *l* from *f*.



Now, when  $a$  and  $b$  converge towards  $p$ , it will also be the case with respect to effect-characters other than  $p$ —say  $x$  and  $y$ —that a change in  $a$  alone would entail a change in  $x$  without a change in  $y$ , and a change in  $b$  alone would entail a change in  $y$  without a change in  $x$ . Processes such as these, from  $a$  to  $x$  and from  $b$  to  $y$ , may be said to be parallel to one another. In the same way, when the processes from  $f$  to  $k$  and  $f$  to  $l$  are diverging, it will also be the case with respect to cause-characters other than  $f$ —say  $m$  and  $n$ —that a change in  $k$  alone would point to a change in  $m$  without a change in  $n$ , and a change in  $l$  alone would point to a change in  $n$  without a change in  $m$ . Again then, processes such as that from  $m$  to  $k$  and from  $n$  to  $l$  may be said to be *parallel* to one another. The additional characters  $\alpha, \beta, \kappa, \lambda$  are introduced in the above diagrams in order to exhibit more fully the significance of parallel, converging and diverging processes.

In these diagrams the horizontal lines from left to right represent the course of time from before to after, so that any vertical line that may be imagined represents simultaneity. On the other hand, the arrows, which are sometimes directed rightwards and sometimes leftwards, indicate—not the temporal opposition of before and after—but the inferential opposition between implying and implied. In speaking of the temporal process from  $ab$  to  $p$  as converging, we mean that when  $a$  and  $b$  are jointly manifested, they both play a part in determining for us what value  $p$  of  $P$  will be manifested. This converging process is represented as preceded by parallel processes  $aa$  and  $\beta b$ , while it will be observed that the parallel processes  $ax$  and  $by$  are contemporaneous with

the converging process from  $ab$  to  $p$ . Similarly in speaking of the temporal process from  $f$  to  $kl$  as diverging, we mean that when  $k$  and  $l$  are jointly manifested, they both play a part in determining for us the value  $f$  of  $F$ . The diverging process is succeeded by parallel processes  $k\kappa, l\lambda$ , while again the parallel processes  $mk, nl$ , are contemporaneous with the diverging processes to  $kl$  from  $f$ . In the interpretation of these diagrams, it is important not to be misled by the irrelevant spatial relations which inevitably enter in diagrammatic representations; for example, the relations of parallelism might falsely suggest such kinematic notions as two particles moving contemporaneously in parallel directions from the points  $\alpha, \beta$ , to the points  $a$  and  $b$ , and then converging towards the point  $p$ . A special application of these diagrams might of course be made to such a kinematic problem; but if it were, the full significance of the diagram could only be exhibited by representing the two particles, whose courses are  $aa$  and  $\beta b$  respectively, not as moving in parallel lines, but as colliding at the moment represented by  $a$  or  $b$ , this collision altering the direction of the particles and accounting for the state of things at the moment represented by  $p$ , which closely follows the moment  $a$  or  $b$ . The parallelism of the lines in the diagram represents the causal or determinative independence of the movements of the particles prior to their collision, and not their spatial directions; for in fact they must have been spatially converging in order to lead to the collision. Another possible interpretation of the metaphor of parallelism which it is important to avoid in the use of these diagrams is that which applies to the case of

psycho-physiological parallelism. There a knowledge of the physiological processes enables us to infer the contemporaneous psychical processes, and conversely; so that parallelism, meaning here epistemic determination, would be represented by two oppositely directed arrows. But in the case which the diagram represents, and which might be illustrated by two billiard balls, the motion of one ball before collision with the other would inferentially determine *no* knowledge whatever of the preceding velocity or direction of the other. Parallelists who tried to combine the notions of determinative dependence and causal independence would have to revise the entire logical account of causality and its connection with epistemic determination.

§ 2. It will give additional significance to our diagrams to further elaborate the kinematic problem. The straight part  $aa$  of the crooked line  $aap$ , might stand for a movement of uniform velocity and constant direction, or rather for the constant retardation due say to the friction etc. of a ball moving on a billiard table. Then the change from  $aa$  to  $ap$  would represent the change of acceleration or retardation which takes place at the moment of geometrical contact with a second ball whose previous course is represented by  $\beta b$  and whose changed acceleration is shown by  $b\beta$ . Furthermore the junction of  $p$  with  $f$  might represent the process of contraction and re-expansion due to the elasticities of the balls. And, lastly,  $k\kappa$  and  $l\lambda$  would exhibit their subsequent rectilinear and causally independent movements.

To show that these diagrams are adaptable to very

different kinds of phenomena, let us now turn to chemistry, and suppose  $a$  to represent the continuously manifested properties of a sample of oxygen, and  $b$  those of a sample of hydrogen. These properties would continue to be manifested without change until some environmental condition brought the two substances into a special spatial and physical connection. If  $a$  and  $b$  represent the moment at which this connection takes place, then at the moment  $ab$  there is initiated a process of chemical combination which leads to the manifestation of new properties—viz. the properties of water—symbolised by  $p$ . The arrow from  $ab$  to  $p$  represents the fact that our knowledge that oxygen and hydrogen are the elements concerned determines for us the knowledge that the properties of water will be manifested in the combination; these properties being jointly determined by the definition of the elements and their spatial relations. As, in the kinematic illustration, so here an appearance of discontinuity or abruptness is presented at the moment when the oxygen and hydrogen combine, and are replaced by the properties of water. Further  $ax$  and  $by$  may be taken to represent the continuance unchanged of the weights of the oxygen and the hydrogen that are combining; these unchanged and independent values continuing to be manifested contemporaneously with the process which ends in the chemical combination. In this way the illustration from chemistry affords an example of the universal principle that along with any converging process of causality, there are always contemporaneous parallel processes—the words parallel and converging being used metaphorically to stand respectively for

causal independence and joint dependence. By bringing into line these examples from chemistry and dynamics we have partially shown in what respects chemical combination and dynamical composition agree, and in what respects they differ. The complete change of motion of the two colliding balls corresponds to the change from the manifestation of the chemical properties of oxygen and hydrogen to that of the properties of water. And the continuance of the weights of the oxygen and hydrogen unchanged during the process of combination corresponds to our conception of the continuance of the masses and resultant momenta of the two balls during the process of collision.

§ 3. The same diagrams serve to illustrate the causal connections between sensation and physical stimulus. We will suppose that the moment  $a$  or  $b$  is that at which the subject becomes aware of two sensations—say of the colours red and blue present simultaneously apart in the field of vision. The physical processes preceding these sensations are represented by  $aa$  and  $\beta b$ , parallelism of the lines standing for the presumed causal independence of these two physical processes. Now, when the sensations  $a$  and  $b$  happen to be incidents in the experience of the same percipient, there will be consequences which would not be entailed if these sensations were experienced by different percipients; these consequences, which we will suppose to be the apprehension of a relation of difference between the two sensations, will be represented by the converging process from  $ab$  to  $p$ . The arrow from  $ab$  to  $p$  illustrates the fact that the apprehension of the relation of difference will be determined by the apprehensions  $ab$  jointly. The contem-

poraneous processes  $ax$ ,  $by$ , might serve to illustrate the continuance unchanged of the apprehensions  $a$  and  $b$ , while the apprehension of their relation of difference is being evolved. But there are many other ways in which the diagram could be interpreted to illustrate the psycho-physical process of sensory stimulation. For instance, if  $a$  and  $b$  stood for the neural processes occurring contemporaneously with the sensations, these would continue to pursue their course, in some respects at least, unaffected by the percipient's cognitive processes of comparison and so on; and in this case the more or less parallel processes  $ax$  and  $by$  would be contemporaneous with the converging process from  $ab$  to  $p$ .

The psycho-physical illustration may be carried further by supposing the points  $p$  and  $f$  to be joined up; this enables us to treat a more complex problem. Thus, if  $a$  and  $b$  stand for different simultaneous occurrences whose relations or connections are apprehended in a synthetic cognitive act (symbolised by  $p$ ), the nature of this perception is causally determined by the nature of  $a$  and  $b$  jointly; hence the arrow passing from  $ab$  to  $p$ . Next taking  $p$  as cause, the effects which follow will be other than those which could have been predicted from a knowledge of the separate processes  $aa$  and  $\beta b$ ; these latter consequences, which occur independently of  $p$ , will be symbolised by the lines  $ax$ ,  $by$ , continued indefinitely in straight or converging directions, of which the course may be said to be parallel to the mental processes and their consequences—the word parallel being used metaphorically to signify determinative independence. For instance, such phenomena as light or heat will engender various physical consequences in the

outer world concurrently with the mental processes and purposes of any individual percipient; and these physical consequences will, in most of their aspects, be independent of psychical process until some new converging process, involving what we may call metaphorically another collision between mind and matter, takes place. Meantime, let us consider in further detail the effects following upon  $p$ , which represented the convergence of physical causes to a psychical effect, these effects being represented by divergent processes in which the causality is from the psychical to the physical. Let us suppose that the perception  $p$  develops, owing to such conditions as the percipient's character and past experience, through processes of cognitive and conative deliberation, into a fiat of the will (symbolised by  $f$ ). The causal process of inner deliberation is represented by a line which may be supposed to join  $p$  to  $f$ . Then giving to the effects of  $f$  the same kind of complexity that we have attributed to the causes of  $p$ ,  $k$  and  $l$  may be taken to represent the diverging manifestations of the fiat  $f$ . The arrow pointing from  $kl$  to  $f$  indicates that the observer can infer from the joint manifestation of  $k$  and  $l$  the character of the fiat  $f$  which caused this manifestation. If then  $k$  and  $l$  represent those phases in the causal process over which the experient has no longer any direct control,  $k$  will develop causally into  $\kappa$ , and  $l$  into  $\lambda$ , along independent lines, such that from  $\kappa$  alone we could infer  $k$  as its cause, and from  $\lambda$  alone we could infer  $l$  as its cause. Thus the two parts of the diagram are joined up, and it is seen how the two independent causal processes  $aa$  and  $\beta b$  may issue—through the intermediary processes from  $p$  to  $f$ —into the two

independent causal processes  $k\kappa$  and  $l\lambda$ . Now knowledge of the laws of causal determination according to which  $a$  evolves into  $\alpha$ , and  $\beta$  into  $b$ , would not by itself enable us to derive the subsequent processes  $\kappa k$  and  $\lambda l$ . Though all these four processes be taken to exhibit the laws according to which physical phenomena are regulated, no mere physical law will account for  $k\kappa$  and  $l\lambda$  as consequences of  $aa$  and  $\beta b$ . To explain these physical consequents of physical antecedents, we must interpolate a converging, an internal, and a diverging process of causal determination whose sphere of operation is the inner consciousness of an individual experient. The joint diagram may be shortly said to represent the alternate action of matter upon mind and mind upon matter.

With regard to *inference* in the case of divergent causal process, while the distinct lines  $fk$ ,  $fl$  indicate that the process may be analysed into two or more distinct part processes, the single arrow pointing backwards from  $kl$  to  $f$  indicates that in general we can infer the determinate value  $f$ , only from a knowledge of both  $k$  and  $l$ , and not from  $k$  alone nor from  $l$  alone. The symbols may, however, represent an analysis from which  $f$  could in certain cases be inferred either from  $k$  alone or from  $l$  alone. To give an instance of inference from joint factors forwards and backwards, we may pursue the course of two billiard balls, forwards from  $aa$  and  $\beta b$  to  $\kappa k$  and  $\lambda l$ , taking  $p$  to represent the forces of compression on both the balls, and  $f$  the forces of expansion on both the balls; then  $p$  would contain the conditions from which we could infer backwards both  $a$  and  $b$ , and  $f$  would contain the conditions from which we could

infer forwards both  $k$  and  $l$ . Equally well we could have inferred forwards from  $aa$  and  $\beta\beta$  the moment and position at which the balls will touch, and from  $k\kappa$  and  $l\lambda$  we can infer backwards the moment and position at which they have touched. But we cannot infer  $k\kappa$  from  $aa$  alone, nor  $aa$  from  $k\kappa$  alone without taking into consideration the movement of the other ball which introduces the converging and diverging processes. Illustrations of the diverging process in which we infer backwards from the conjunction of two or more effects, the nature of the cause, are well furnished by the case of symptoms. Thus in medical diagnosis it is often impossible to infer the nature of a specific disease from any of the symptoms separately, and it is therefore necessary to join different symptoms in order to infer their cause. Similarly the effects of different emotions such as anger and fear, as manifested in bodily disturbances, partially agree and partially differ; hence a number of factors would have to be noted in order to infer in any given case whether the cause of the bodily disturbances was fear or anger. Purposive action affords a peculiarly interesting example of our analysis of causality into converging and diverging processes. Such action may in general be defined as involving a divergent process issuing from a thought of an end, followed by a convergent process in the outer environment in which this same end, previously represented in thought, is actualised in fact.

§ 4. We will now consider certain more complicated cases of causal process which exhibit convergent, divergent and parallel processes contemporaneously. For illustration we will take two particles whose movements

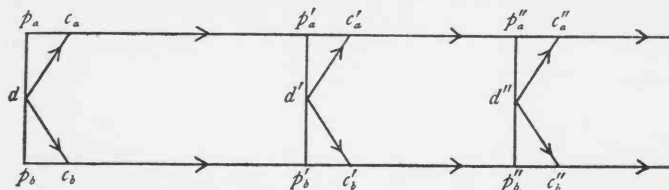
are determined under some such law as that of gravity. Let  $a$  be one particle, and let  $p_a$  and  $p'_a$  be two positions successively occupied by  $a$  at two moments separated by an assigned interval of time. We cannot infer  $p'_a$  from  $p_a$  alone, but only from knowledge of  $p_a$  jointly with the change of motion which  $a$  is undergoing when at position  $p$ ; if this change of motion be symbolised by  $c_a$ , we may then speak of  $p'_a$  as determined jointly by  $p_a$  and  $c_a$ . Adapting our previous diagram to this relation of causality, we have the following:

$$\frac{p_a}{c_a} \longrightarrow p'_a$$

where  $p_a$ ,  $c_a$ ,  $p'_a$  take the place of  $a$ ,  $b$ ,  $p$  respectively, and the figure represents a converging process. Before introducing a second particle  $b$ , we will simplify the above diagram by bringing  $c_a$  close up to  $p_a$ , understanding by this juxtaposition literal simultaneity, and then join  $p_a$ ,  $c_a$  to  $p'_a$  by the horizontal arrow. The motion of particle  $b$  is similarly represented by symbols in the second diagram. Now when  $a$  is at position  $p_a$ , and  $b$  is simultaneously at position  $p_b$ , the distance between  $p_a$  and  $p_b$  is a determining condition from which we can infer the change of motion of both  $a$  and  $b$ , under the law which we have assumed to be that of gravity: this relation of distance, therefore, stands as a cause-condition diverging into the two effects  $c_a$  and  $c_b$ . The process is exhibited in the following diagram, which also illustrates subsequent positions of the particles subject to the same type of causality. Here, at the first moment, the distance  $d$  between  $p_a$  and  $p_b$  determines divergently the changes of motion  $c_a$  and  $c_b$ , while the position  $p_a$  and the change  $c_a$  determine convergently the position

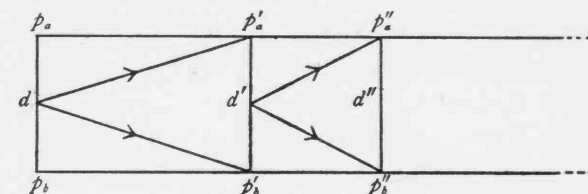


$p'_a$ ; similarly the position  $p_b$  and the change  $c_b$  determine convergently the position  $p'_b$ . It is also important to note that the converging process  $p_a c_a$  to  $p'_a$  is determinatively parallel to, that is independent of, the converging process from  $p_b c_b$  to  $p'_b$ . The same relations are exhibited at the next moment considered in relation to the third moment, where the dashes serve to distinguish the

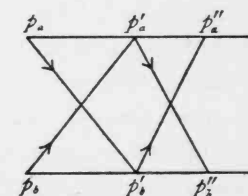


several moments. Between each of the moments separated in the diagram, we must suppose an indefinite number of the same configurations following one another continuously. And since, at any instance, the distance  $d$  is a causal factor common to the movements of  $a$  and of  $b$ , the movements from  $p_a$  to  $p'_a$  and from  $p_b$  to  $p'_b$  are not properly called parallel in the determinative sense, when an appreciable interval of time has elapsed. It must also be remembered that the positions and juxtaposed changes of motion are to be conceived as literally simultaneous and not as continuously successive. We have spoken of  $p_a$  and  $c_a$  as jointly determining  $p'_a$ ; we may equally speak of  $p_a$  and  $p'_a$  as jointly determining  $c_a$ ; and this illustrates the commutative principle for what has been called a prime dependency. The three values  $p_a$ ,  $c_a$ ,  $p'_a$  may be said to constitute a kinematic prime dependency: it is actually by the observation of  $p_a$  and  $p'_a$  that we infer  $c_a$ , while we regard  $p_a$  and  $c_a$  as causally determining  $p'_a$ . Thus a knowledge of  $p_a$ ,  $p'_a$

and  $p_b$ ,  $p'_b$ , could take the place of the knowledge  $p_a$ ,  $c_a$ , and  $p_b$ ,  $c_b$ , in determining the whole course of the action both backwards and forwards. The diagram could then be simplified by omitting the symbols for change of motion on a principle analogous to the triangle of forces, so that a single arrow from  $d$  to  $p'_a$  will replace the two arrows from  $d$  to  $c_a$  and  $c_a$  to  $p'_a$ , thus:



To indicate the continuity of the process, we may still further condense the diagram. In the former of these two diagrams  $p_a$  and  $d$  determine convergently  $p'_a$ ; while  $p_b$  and  $d$  convergently determine  $p'_b$ , these two processes being themselves divergent. In the latter of the two diagrams we represent  $p_a$  and  $p_b$  as convergently determining both  $p'_a$  and  $p'_b$ , while  $p'_a$  and  $p'_b$  are divergently determined by  $p_a$  and  $p_b$ . A slightly different interpretation of the symbols will elucidate this apparent contradiction. If the symbols  $p$ , instead of representing the mere geometric notion of position, be interpreted kinetically to include position and determinate motor tendency, the relation  $d$  of  $b$  to  $a$  will then be conceived as a causal condition modifying the motor tendency and thus effecting an actual motion other than that which the tendency by itself would have produced.



The cause conceived in this way has no effect peculiar to itself, but modifies what would otherwise have been the determinate effect. The phenomenon  $b$ , when in the relation  $d$  to  $a$ , which thus modifies the process of  $a$ , is conceived as agent relatively to  $a$ , and  $a$  is conceived as patient relatively to  $b$ . In philosophical terminology we speak of the unmodified process of  $a$  as illustrating immanent causality, and the modifying influence of  $b$  upon  $a$  as illustrating transeunt causality. Apart from these disputable terms, the consideration with which we are here concerned is that in order to define the nature of the effect which the relation  $d$  of  $b$  has upon  $a$ , it is necessary to introduce reference to the trend or motor tendency which  $a$  is manifesting at the moment when  $b$  influences it. This serves to illustrate the point that the idea of change is complex, and needs to be carefully examined: it does not mean simply a difference in the state of  $a$  at one moment as compared with its state at a subsequent moment, but it means a difference between the state into which  $a$  passes under the operation of an external causal agency, such as its determining relation  $d$  to  $b$ , as compared with the state into which  $a$  would have passed by its own agency.

§ 5. This complex form of causality may be illustrated from psycho-physical process as well as from dynamics. In this case the symbol  $a$  in the condensed diagram will stand for the mental side of such a process, and the symbol  $b$  for the physical side; the letters  $p$  representing not statically defined states, but motor trends. Thus if  $p_a$  represents the course which a sensation process is taking at any moment independently of any physical process such as  $b$ , the physical stimulus  $p_b$ , as soon as

it begins to operate, will affect this sensational trend, and determine it in the form  $p'_a$ , which is a modification of what  $p_a$  would have become *apart* from the stimulus. Now, if the subject is what may be described as *inactive* with regard to the further course of his sensations, the arrow in our diagram will be drawn only from the line of  $b$  to the line of  $a$ , and the arrows in the opposite direction may be omitted. The diagram would then represent a state in which the sequence of sensations was wholly determined by the course which the physical or physiological processes assume under purely physical laws, and where there was no reaction from the side of the psychical to the side of the physical. But now let us suppose that the subject is active and takes a share in determining the course of his sensations. It must be admitted that such active determination by the subject is not a direct causal determinant; and the facts are illustrated by the diagram with all the arrows inserted. Thus we shall define  $p_a$  not as a mere passively received sensation, but as a cognition, having in it an element determined by the nature of the stimulus  $p_b$ , and besides this, other related cognitive elements more or less complicated according to the degree of intelligence of the supposed subject. The arrow, therefore, from  $p_a$  to  $p'_b$  indicates that  $p_a$ , defined—not as a mere sensation—but as a cognition, causally determines  $p'_b$  in the same way as  $p_b$  was previously shown to determine  $p'_a$ ; that is,  $p_a$  does not bring  $p'_b$  into existence, but it determines the actual form assumed by the physiological process  $b$ , in the sense of modifying the form  $p_b$  would have taken, apart from the active determination of this cognition. The whole process is descriptively condensed in the phrase

that the physiological course and the sensational course reciprocally determine one another; neither would be what it actually is, if the influence of the other had been non-determinative. This condensed description, however, is more accurately analysed into an alternate process from the side of mentality to the physiological and reversely from the physiological to the side of mentality. The mentality involved is not purely passive sensation, but actively determinative cognition, involving (at least) what psychologists call attention; and in cases of a higher level of intelligence, more or less co-ordinated purpose. The process indeed, which the subject cognises, is itself mental, and must not be confused with the course of the physiological changes themselves, of which he is wholly unaware; his attention is actually directed to the changes in the sensational experience of which he is retrospectively and more or less prospectively aware.

The condensed diagram interpreted so far to apply to the causal interrelations between a merely physiological process on the one side and active mentality on the other, can be used to illustrate a wider range of interaction between mind and matter, which shall include operations on the external environment. In this application the line  $b$  no longer represents purely physiological processes, but includes processes in the external physical world. Here again the important consideration is that the purposive thought  $p_b$  does not bring into existence the physical phenomenon  $p'_b$ , but it determines the phase of  $b$  to be otherwise than what  $p$  would have become under the determination of purely physical causality. Of course the mode in which the

course of  $p_b$  passes into the phase  $p'_b$  is not adequately represented by a single straight line, and in the diagram a very complicated process is artificially condensed. In fact there will be sections of the physical process that are left—uninfluenced by the subject—to follow a course determined by purely physical causality; and to represent such sections, the arrows from the side of mentality to the physical side should properly be omitted. Changes of this kind may be observed by the subject, and his observation of the phase into which the process has passed may determine him to initiate a new interfering or controlling operation which will again modify the further course of the physical process. The moment when this observation occurs will be marked by an arrow from the physical side to the side of mentality, since it is the nature of the physical occurrence which determines the content of the predicative cognition on the part of the observer. In its turn, this cognition will operate on the other variously modifiable inner processes of the mind, and determine a corresponding reaction, modifying the physiological as well as the physical course of things; and these changes will be marked by an arrow from the side of mentality to that of the external and physical. This arrow is again a condensed representation of converging process; for the phase  $p'_b$  determined from the side of mentality, is due jointly to the just preceding observation, taken along with the appreciation of  $p'_b$  as a phase in the progressively attained purpose, as well as the knowledge of the activities needed for furthering this attainment.

Finally we may close the exhibition of the entire purposive process by a set of lines converging upon

that terminal phase of actualised experience which denotes the realisation of the end corresponding to the thought of the end from which at the beginning the initial processes diverged. Thus, the scheme of purposive causality begins and ends as a phase in the consciousness of the same individual thinker or actor; while the intermediate or instrumental phases are incidents in the world of physical phenomena, some of which are within the organism and nervous system, and thus in the most direct causal contact with the thinker's feelings, thoughts, and powers of causal determination.

## CHAPTER XI

TEMPORAL AND SPATIAL RELATIONS  
INVOLVED IN CAUSALITY

§ 1. THE general discussion of connectional determination entails consideration of the spatio-temporal relations amongst phenomena in terms of which occurrences are represented as bound together in a unity of connection. Thus, the bare formula '*abcde* determines *p*,' where the symbols stand for the characterising adjectives of occurrences, is a merely abstract expression of the causal principle, inasmuch as no reference is explicitly made to the spatio-temporal nexus (as it may be termed) under which the manifestations of these characters take place. When the event characterised as *abcde* is said connectionally to determine an event characterised as *p*, these events have spatio-temporal extension as also spatio-temporal relations one with another. The manifestations *a, b, c, d, e* will be termed *occurrents* severally, and their conjoint manifestation will be termed an *event*. These occurrents are *several*, because the determinate character of each comes under its own determinable *A, B, C, D, E* respectively. On the other hand, *events* are *several*, because each has its own distinct spatio-temporal boundary. The *extension* of an event allows us to speak of any event as containing spatio-temporally distinguishable parts, which are themselves events. On the other hand, the occurrents *a, b, c, d, e* are not parts but constituents of the event, for they all occupy one and the same spatio-temporal position defined by the boundary

of the event. Furthermore, connectional determination signifies that the *position* of the manifestation of *p* relatively to that of *abcde*, as also its determinate *character* *p*, is determined jointly by the characters of the coincident manifestations *a, b, c, d, e*.

So far we have treated the notion of nexus as concerned solely with spatio-temporal relations. But the above account must be amplified and, in a sense, partially amended by noting that every occurrent must be referred to its own proper continuant. Thus, to the contrast between occupying the same or different positions must be added that between being referred to the same or to different continuants. In fact, distinctions of position must be understood metaphorically to extend to distinctions of continuant-reference. And, for similar reasons, determinables must be distinguished according as their determinate values characterise manifestations referred to this or to that continuant.

§ 2. We will now examine the general notion of Order. Order is predicated of terms which, for convenience of figurative representation, may be called Points, and when Points are in an Order the collection is called a Series. Taking any three points whatever in a Series, these may be said to be in a determined order when there is some assignable principle according to which one of the three points is to be regarded as 'between' the other two. Thus the three notions of a Series, or Order, and of 'betweenness,' mutually involve one another. These remarks apply equally to what may be called a discrete Series, or to a continuous Series—two types of Order which may be distinguished as follows: a continuous series is such

that between *any* two non-identical Points, there is a Point non-identical with both. The series of integers 1, 2, 3, 4, ..., on the other hand, illustrates a discrete series, for, between the two integers 3 and 4 for instance, there is no integer; again the dots after 4 may also be taken to illustrate a discrete series, for between the first and second there is no dot, as also between the second and third, and so on. A discrete Series is, in fact, always figuratively represented by dots spatially separated along a line; and the fact that it is natural to name these points by the ordinal numbers shows that a series of integers most naturally illustrates a discrete Series. Similarly it is natural figuratively to represent a *continuous* Series by a line, containing points such that between any two points there is in the Series a point different from both. When a line is regarded as the boundary between two contiguous areas of a surface, it enables us to conceive of a *discrete* series of areas; thus we can count one by one a series of contiguous areas by mentally representing the lineal boundary common to any two; but in such case the entire surface is to be described as *continuous*, for between any two lineal boundaries, there is in this surface a lineal boundary different from both. The surface itself may be regarded as the boundary dividing a region into a discrete series of parts; but again in this case, the entire region is to be described as continuous; for between any two areal boundaries within the region, there is an areal boundary different from both. We are thus led to distinguish between the parts of a whole, and the boundaries between these parts. The parts of a line are lines, the parts of an area are areas, the parts of a region are



regions; but the boundary between contiguous parts of a line is a point, and the boundary between contiguous parts of an area is a line, and the boundary between contiguous parts of a region is an area or surface. The parts of a whole, therefore, are homogeneous with one another and with the whole; but as we noted when contrasting extensive with extensional wholes<sup>1</sup>, the boundaries between two contiguous parts are always of one lower order of dimensions than the parts.

Our illustrations of discreteness and continuity have so far been taken solely from Space; but the notions are equally applicable to Time. Thus Time is conceived as of one dimension, and is composed of parts which are periods, the boundary between two contiguous periods being called an *instant*. A period of time, therefore, corresponds to a *line*, and an instant corresponds to a *point*; the period-parts of a period will then constitute a discrete Series, and the instants—i.e. the boundaries between two contiguous period-parts—will constitute a continuous Series. The above application of the term 'discrete' to contiguous parts of a whole might be criticised as being incompatible with its original application to separated points. But it must be noted that the notion of discreteness does not imply *factual* separation, but only separation *in thought*. When we think of a boundary between two contiguous parts, we are mentally separating those parts, without predicating any factual separation; in this sense we may always say that a whole can be conceived as a sum of its discrete parts, whether the whole is such that it can be said to contain contiguous parts, i.e. parts having a common

<sup>1</sup> Part II, Chapter VII, § 8.

boundary, or not. In the former case it is usual to call the whole *continuous*, and unusual to allow of its being also called *discrete*; but it appears to me that the notion of continuity is derived from that of contiguity, and that the definition of contiguity involves the notion of a common boundary between two parts; hence, for the notion of a continuous whole, I prefer to substitute the notion of a whole consisting of parts, whose boundaries constitute a continuous Series.

§ 3. Having contrasted continuity with discreteness, we will now examine another meaning frequently attached to the word *continuous*, which may, I think, be conveniently contrasted with *discontinuous*. The term *discrete*, as hitherto explained, applies to a single variable whose variations are not considered in connection with the variations of other variables. We have now to consider so-called *correlated* variables, the variation of one of which is dependent on that of the other according to some assignable formula. In this case we shall find that while the independent variable is continuous, the changes of the other variable correlated with this continuous series may be either continuous, or (as it may be described) *discontinuous*. The dependent variable will be said to vary continuously when whatever section of its actual variation is considered, every possible value intermediate between those assumed at the beginning and end of the section are actually represented; it will be said to vary discontinuously when within *some* section of its actual variation there are intermediate values which are not represented. The most familiar instance of this kind of correlated discontinuity or continuity are those in which Time is the independent

variable. The actual variations of Time are continuous in the first sense of the term, and every possible value is actually represented; but there may be gaps in the variations of the variable which depends, according to some formula, upon the variation of Time. A simple illustration is that of acceleration: thus if a body is moving on a rough horizontal table until it falls over the edge, then at the instant when it begins to fall there is discontinuity in the change of acceleration. While it is moving horizontally its movement is subject to the retardation of friction operating horizontally; when it is falling, on the other hand, its movement is subject to the acceleration of gravity which operates vertically. There must, therefore, be an instant in which the acceleration or retardation changes from one value to another with the omission of all the possible values intermediate between the horizontal retardation and the vertical acceleration. The acceleration in such a case varies discontinuously, but not so the *velocity*; for every possible velocity intermediate between the rate of movement of the body on the table and the rate when it is beginning to fall, is assumed by the body during the intermediate time; for the body does not fall vertically, but—neglecting the resistance of the air—along a parabola.

Now according to a prevalent view in philosophy, the theory that *all change is continuous* is intuitively axiomatic. But the change of acceleration in the above case would seem to contradict this theory; although physicists do, as a matter of fact, hold that the change of velocity *is* continuous. The explanation of the apparent contradictions to the theory is to be found in the discontinuity of the physical processes corre-

lated with the continuous series either of Time or of Space. For example, some of the surfaces in Space are boundaries separating contiguous bodies of quite different characters, such as solid and gaseous; here then, discontinuity holds of the dependent variable 'physical character' as determined by the independent variable 'spatial position.' Thus Time and Space are conceived as continuous in the first sense of the term, but the variations correlated with these independent variables are frequently discontinuous; and when this is the case discontinuity of the variations correlated with the variation of Time is explained by the discontinuity of the variations correlated with the variation of Space.

It should be observed that when speaking of change in the surface of a body, one part of which is solid and another part liquid, or one part rough and another part smooth, or one part red and another part green, the word change is applied to variations correlated with variation of Space instead of Time. We are apt to regard the words variation and change as synonymous, but it is very important to restrict the term variation to uncorrelated differences, and to apply the term *change* to differences correlated either with differences of Time or with differences of Space. This distinction affords some explanation of the rather vague statement of philosophers that causal process is continuous: a causal process is a process of change, and as such is correlated with a variation of Time or of Space, Time and Space being admittedly continuous; but the variations themselves in the changing variable correlated with Time or Space may be discontinuous. To elucidate this seemingly paradoxical notion of discontinuous variation, let us

imagine a process of change the beginning and end of which are dated by the numbers 1 and 3. If the symbol *a* represents the determinate character dated at 1, and the symbol *f* the determinate character manifested at date 3—*a* and *f* denoting different determinates under the same determinable, whose determinates *a*, *b*, *c*, *d*, *e*, *f*, ... can be ranged in an order depending upon a comparison of degrees of difference—then the correlated discontinuity is illustrated by assuming that during the period from 1 to 3, the stretch from *c* to *e* (containing *d*) is *not* manifested. Through the period from date 1 to date 2, we will say, the manifestation changes continuously from the character *a* to the character *c*; and through the period from date 2 to date 3, it changes from *e* to *f* also continuously. At no instant within the period from 1 to 3 is the character *d* manifested. At the terminal phase belonging to the period 1 to 2, the manifestation must be characterised as *c*; but at the initial phase belonging to the period 2 to 3, it must be characterised as *e*. As so regarded the dates of these two phases cannot be identified; hence we cannot speak of *any* determinate character as being manifested *at* the date 2. On this ground therefore, at the instant in question, discontinuity of variation must be attributed to the process. A similar illustration, with Space instead of Time as the continuous variable, is afforded by supposing the surface of a table divided into two contiguous parts, one of which is entirely red, and the other entirely green, and considering the colour of the line which is the common boundary of these two parts. This question can perhaps hardly be said to involve a paradox, for colour characterises

a *surface* and the surface-parts of a surface, it does not characterise any line which is contained in—but is not a part of—the surface. If this solution of the paradox be accepted, by the same method the paradox involved in correlation with Time may be removed; for, just as the parts of a surface which are two-dimensional as is the whole surface, must be distinguished from the boundary or separating lines which are contained in, but are not parts of, the surface; so the parts of Time which are of *one* dimension (as is the whole Time) must be distinguished from the instants, which are of *no* dimension, and which are contained in (but are not parts of) Time. And, just as colour was said to characterise the surface-parts of the table and not the lines contained in these surface-parts, so acceleration must be taken to characterise a movement occupying a certain part of Time, and not a *position* of the moving body which is correlated with an instant; for an instant is contained in, but is not part of, the period of Time to which the process of change is referred.

§ 4. Correlated continuity is probably always exhibited by immanent processes, i.e. these are non-discontinuous in their correlation with Time. The philosophical problem arises when transeunt causality is introduced. It may be stated thus: How is it that at a certain moment of time, two separate processes which have been immanently determined previously to this moment of time, cease to be for the subsequent time determined by merely immanent causality? This can only be explained by supposing some kind of connectional determination; i.e. if *S* and *T* represent the two immanent processes, in order to account for transeunt action taking

place at one instant of time rather than at another, we must suppose some kind of connection between the manifestations of *S* and the manifestations of *T*; and the postulate is constructed that where there is transeunt action of *T* upon *S*, there is involved in the formula of determination a different mode of connection from that of Time. To this form of connection we give the name spatial, because in the case of physically determined phenomena transeunt causality always does operate in Space. Defined relations of spatial connection enter only in the formulae of interphysical causality, whereas every kind of causal formula involves defined relations of temporal connection. In the chapter on transeunt and immanent causality I have suggested that, for interpsychical causality, what takes the place of spatial connection is the attachment of both feeling and cognition to the same object; including, under the term feeling, conation in its two forms of attraction and repulsion felt with greater or less intensity towards experiences perceptually or imaginatively apprehended; and under the term cognition, including the variations in degrees of determinateness, as well as of *content*, according as the experience is thought of as having this or that character. Thus in my view the formula of interpsychical causality, introducing variable relations of feeling and cognition as causal determinants, takes the place of variable spatial relations as causal determinants in interphysical causality. For simplification of this exposition I have supposed the transeunt action to operate *ex abrupto* so that the instant of time at which it is dated can be determinately assigned. But our account of transeunt action must be extended to the

cases where *some* spatial relation, which either remains unchanged or alters in the course of time, continually subsists between the manifestations of *S* and *T*, and where transeunt action is therefore temporally continuous instead of being instantaneous.

§ 5. We may now turn to the specific topic of this chapter, and consider the temporal relation of cause to effect which is commonly said to be that of before to after. In the first place it must be pointed out that manifestations cannot be related merely under the form of before to after, but must always, in addition, be regarded as manifestations of the same continuant-entity, whose nature is expressed in the formula according to which the preceding manifestations determine the succeeding. And, in the second place, parallel with the temporal order amongst the manifestations of a continuant, we have to consider the spatial order amongst the manifestations of an occupant. In somewhat figurative language we may conceive of an occupant as manifesting itself in a succession of regions which form rings from the more inner to the more outer, separated by concentric boundaries. The relations of the inner to the outer manifestations of an occupant are analogous to the relations of the preceding to the succeeding manifestations of a continuant. An adequate knowledge of the immanent nature of the entity would enable us to infer equally from the preceding to the succeeding, as from the succeeding to the preceding, in the case of a continuant; and from the inner to the outer, as from the outer to the inner in the case of an occupant. Thus reference to changing and spreading characters to the same continuant or occupant is the basal principle underlying

causality. Now in spite of this possibility of reciprocal inference, we nevertheless regard the preceding as objectively determining the succeeding, as well as the inner as objectively determining the outer; and never reversely, the succeeding as objectively determining the preceding, and the outer as objectively determining the inner. The explanation of this refusal to reverse the order of objective determination in the temporal and spatial manifestations of continuants and occupants requires us, I think, to pass from immanent to transeunt causality. Thus, at a certain moment of time, an immanent process of causality may be broken in upon from without by an influence which modifies the succeeding manifestations, so that these are different from what they would have been under the uninterrupted course of immanent process. So while the manifestations which preceded the interruption can be determined from the mere knowledge of the formula of immanent causality, after the interruption the relation of the succeeding to the preceding is objectively differentiated from that of the preceding to the succeeding. It is to be noted that the temporal relation involved in transeunt causality itself, is neither that of succeeding nor of preceding, but of simultaneity. Analogously, in the case of the occupant, the reason why we regard the inner manifestations as objectively determining the outer, and not reversely, is explained by the introduction of transeunt causality. Just as, in time, we can take the boundary between the preceding and succeeding phases, and show that when a cause from without operates at this boundary, the succeeding phases are immediately modified; so we may take the surface of an occupant as the boundary at which a

cause from without operates, and show that it immediately modifies the outer and not the inner state of the occupant. Simple illustrations of this principle are afforded by an extensible string, or a compressible fluid: when a string is subjected to an equal tension at both ends, the transeunt causality is from the extremes to the centre, while the immanent causality, which reacts, is from the centre to the extremes; or again, when a fluid is subjected to equal pressure throughout its surface, the transeunt causality is from the outer surface to the inner, while the immanent causality with which the fluid reacts is from the inner to the outer.

§ 6. The above account requires some explanatory modification; for in all such cases as those we are considering, manifestations of an occupant which are *actual* over certain regions of space, at any given time, are *potential* over other regions; and similarly, manifestations of a continuant which are actual throughout certain periods of time, are potential throughout the other periods. This point is illustrated with peculiar significance in psychology, where periods of apparent abeyance in consciousness of the familiar phenomena of association illustrate in the extremest sense the potentiality for manifestations. The occasions when this potentiality is converted into an actuality are when an experience breaks in upon the previous current of thought and operates transeuntly in modifying the subsequent processes. In a precisely similar way, the occasions when the potentiality of a body for exerting pressure or sustaining tension is converted into actuality, are those when it comes into transeunt contact with a foreign body which modifies its subsequent states. In any



case of this kind we may distinguish those manifestations which are modified by the transeunt action from those which could have been determined without knowledge of such action. It will be found that the unmodified manifestations of the continuant are related to the modified as earlier to later, and of the occupant as inner to outer. Thus to take the occupant, for example, when a foreign body attracts a given body as a whole, it does not affect the internal motions of its parts, represented by temperature, chemical constitution, inner strains and stresses, etc., but only its situation relatively to other bodies, and these may be properly called outer manifestations relatively to the inner and immanent processes of the body. Contrasting an illustration of this kind with such transeunt action as the application of heat to a gas, the transeunt causality in the latter case appears to produce effects in the inner region as well as the outer occupied by the substance; but this is because the gas does not in truth constitute a unit-entity, and must be broken up into parts before we can apply with significance the distinction between the immanent and the transeunt. From the point of view of mechanical and thermal analysis the parts into which the gas must be broken up are molecules whose only inner and immanent manifestations are chemical. The application of heat affects the actions between the molecules themselves, as represented by their relative movements and mutual pressures, and these illustrate transeunt causality, the chemical or inner processes of the molecules being left unaffected to follow their own immanent course. The case of the gas, then, when properly analysed, is a further illustration of the principle that the transeunt

processes modify the outer and the later manifestations without affecting the inner and the earlier; and that between the transeunt cause and the transeunt effect there is temporal simultaneity and spatial coincidence.

§ 7. The above illustration of transeunt and immanent action suggests a third kind of causality which requires separate consideration, viz. that involved in the compression of a compressible solid, as distinguished from the compression of a gas. Here the correlated conceptions of stress and strain are properly applied; a liquid or a solid when it is unnaturally compressed, exerts a force of expansion which decreases from a certain maximum to the minimum zero, as the compression is allowed to decrease. And a solid or extensible string, when it is unnaturally extended exerts a force of contraction which decreases from a certain maximum limit to the minimum zero as the extension is allowed to decrease. It will help us to understand the nature of the force of tension illustrated by the string if we contrast it with the force of attraction; for, while between two attracting bodies the force of approach is stronger the nearer they are to one another, between two parts, say, of a string the force of approach is stronger the further they have been pulled from one another. Now to understand the type of causality operating in the case of the compressed solid, we may mentally divide the volume which it occupies into an inner core and an outer ring. The effect of the pressure operating from the foreign force is to unnaturally contract the volume occupied by the inner core, causing an outward pressure upon the inner side of the outer ring. Apart from the outward pressure, the effect of the inward pressure would be manifested

in the restoration of the outer ring to its natural size; we may therefore properly take the contracted condition of the inner core as the immanent cause of the subsequent expansion of the outer ring. This form of causality illustrates the same type of analogy between temporal and spatial factors as we have already noted, the inner in space corresponding to the earlier in time, and the outer in space corresponding to the later in time, while the causality operating at the common boundary between the inner core and the outer ring corresponds to the moment of time at which the condition of the inner core influences the condition of the outer ring. If then immanent causality alone were involved, our knowledge of the shape and size of the inner core would determine for us a knowledge of the subsequent and contiguous shape and size assumed by the outer ring. But when the transeunt causality from the foreign force is brought into consideration, the subsequent and contiguous shape and size assumed by the outer ring is modified. Having divided at an arbitrary surface the inner core from the outer ring, we must make a correspondingly arbitrary division in time between the earlier and the later states of the body. Considering the solid body alone, the inner core is first under a pressure dependent upon its unnatural shape and size, and the subjection of the outer ring to the foreign compressing force of pressure is later in time. So the inner surface of the outer ring at the earlier stage is pressing outwards, and the outer surface of this outer ring at the later stage is pressing inwards. Hence the pressure at the inner surface at the earlier stage represents that part of the process (determined solely by immanent conditions) which is unmodi-

fied by transeunt action; while the pressure at the outer surface at the later stage represents that part of the process which is modified by transeunt action. The case of the extensible string is capable of precisely similar analysis; so also is that of the varying temperature of gas enclosed in an envelope. In all these cases, the immanent tendencies operate in the direction of an assignable form of equilibrium, and by dividing the whole process into temporal and spatial parts corresponding to one another we shall always find, by taking the earlier stage to correspond with the inner region, and the later stage with the outer region, that the former represents the part of the process unmodified, and the latter the part of the process modified by transeunt action.

A failure in the analogy between Space and Time hitherto unnoted, may here be pointed out. Whereas the dating of a process is absolute, in the sense that it is independent of the continuant to which the process refers, the locating of a process as being relatively inner or outer is not absolute, for what is relatively inner to one occupant is relatively outer to another. To establish the required analogy, it would be necessary to conceive that, of two temporally distinguished parts of a process extending through Time, that which is earlier when referred to one continuant is later when referred to another; just as, of two spatially distinguished parts of a process extending through Space, that which is inner when referred to one occupant is outer when referred to another.

## APPENDIX ON EDUCATION

§ 1. IN the problem before us we shall be concerned with a certain adjectival determinable  $P$  which has  $a$  determinate values— $p_1, p_2, \dots p_a$ —and shall proceed to consider  $M$  instances, each of which is characterised by one or other of these  $a$  determinate characters.

Any actual set of occurrences of length  $M$  will exhibit a certain proportion among the  $a$  determinate characters;— $m_1$  occurrences of  $p_1, m_2$  of  $p_2 \dots m_a$  of  $p_a$  (say), where

$$m_1 + m_2 + \dots + m_a = M.$$

The proportion  $m_1 : m_2 : \dots : m_a$  exhibited in  $M$  occurrences will be denoted by  $\mu$ .

These occurrences also will be presented in a definite order.

The order in which the occurrences exhibiting the proportion  $\mu$  are presented will be denoted by  $\vec{\mu}$ .

The following two elementary arithmetical formulae will be required:

### (1) *Combination-formula.*

The number of integral solutions of the equation

$$m_1 + m_2 + \dots + m_a = M,$$

i.e. the number of values that  $\mu$  may assume in  $M$  occurrences, is

$$\frac{a(a+1) \dots (a+M-1)}{M!}.$$

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### (2) *Permutation-formula.*

The number of different orders  $\vec{\mu}$ , in which a given proportion  $m_1 : m_2 : \dots : m_a$  may be presented in  $M$  occurrences, is

$$\frac{M!}{m_1! m_2! \dots m_a!}.$$

§ 2. Probability is a magnitude to be attached to any possibly true or possibly false proposition; not, however, to the proposition in and for itself, but in reference to another proposition the truth of which is supposed to be known. For example, the probability of the proposition that 'The next throw of a certain coin will yield head' may have its value assigned by the knowledge that 'It will yield either head or tail.' The value of the probability as so determined is not necessarily the same as that determined by the knowledge that 'The previous throws of the coin have presented heads and tails with a certain frequency.' The proposition to which the probability attaches will be conveniently termed the *proposal*; and the proposition to which the probability refers as that whose truth is supposed to be known will be conveniently termed the *supposal*.

Furthermore, in order that the notion of probability shall have significance, it is requisite that the proposition standing as supposal *shall not be known to be false*. Using the notation adopted by Mr J. M. Keynes, which introduces the solidus:

$p/s$  symbolises the probability of the proposal  $p$  as depending upon or referred to the supposal  $s$ .

The notation  $p/s$  may be read ' $p$  upon  $s$ ' or ' $p$  given  $s$ .'

The maximum limiting magnitude of  $p/s$  is certitude: viz. when the truth of  $p$  is implied by  $s$ . Its minimum limiting magnitude is contra-certitude: viz. when the falsity of  $p$  is implied by  $s$ . Since probability-values are signless, the minimum value (contra-certitude) must always be represented as *zero*; and, since certitude is the maximum probability-value, all other probability-values are (proper) fractions of certitude. It is, in fact, usual to express probability-values as pure fractions, such as  $\frac{1}{2}$  or  $\frac{2}{3}$ ; and to express certitude by unity. But this representation is logically false, and should only be permitted as a convenient abbreviation.

In estimating the probability of  $p$  as depending on the specific knowledge  $s$  it is essential that  $s$  should represent the *whole* of the supposed knowledge, relevant to the case. Briefly, the dependence indicated by the equation  $p/s = \frac{2}{3}$  of certitude (say), when expressed as an implication, means:

If  $s$  *alone* were known, then the probability of  $p$  would be  $\frac{2}{3}$  of certitude.

If  $t$  also were known,  $p/st$  would not necessarily be the same.

In this respect, the relation of dependence for probability is to be contrasted with the relation of implication. Thus

' $p$  is implied by  $s$ ' corresponds to  $p/s = \text{certitude}$ .

Now, if ' $p$  is implied by  $s$ ,' then also ' $p$  is implied by  $st$ ' and hence  $p/st$  also = certitude.

In other words, additional supposed knowledge cannot alter the degree of probability of any proposition known to be true or to be false, but it may always alter the degree of probability of a proposition not known to be true or to be false.

§ 3. Two axioms are required for the working of the probability-calculus: viz. the additive and the multiplicative. With the notation above explained, these axioms may be formulated as follows:

*Additive axiom:*

If  $p$  and  $q$  are known to be not both true, then

$$(p \text{ or } q)/h = p/h + q/h.$$

*Multiplicative axiom:*

If  $p$  is not known to be false, then

$$(p \text{ and } q)/h = p/h \times q/(p \text{ and } h).$$

When such symbols as  $p, q$  stand for propositions, the conjunctive ' $p$  and  $q$ ' will be abbreviated into  $pq$ . But, when  $x, y$  (say) stand for quantities, then  $x \cdot y$  or  $xy$  will mean ' $x \times y$ .' On the other hand, ' $p$  or  $q$ ' should never be written ' $p + q$ '; nor should  $p/q$  be written  $p \div q$  or  $\frac{p}{q}$  (in spite of certain analogies).

Thus the formula for multiplication may be written

$$pq/h = p/h \times q/ph.$$

§ 4. The following corollaries will be required in the sequel:

COR. 1. If  $p_1$  or  $p_2$  or ...  $p_r \equiv p$ , where  $p_1, p_2, \dots, p_r$  are co-disjunct, then, by additive axiom,

$$p/h = p_1/h + p_2/h + \dots + p_r/h.$$

COR. 2. If, further,

$$p_1/h = p_2/h = \dots = p_r/h,$$

then each of these =  $\frac{p/h}{r}$ .

COR. 3. If ' $q$  implies  $p$ ,' i.e.  $q \equiv p$  and  $q$ , then

$$q/h = (p \text{ and } q)/h = p/h \times q/ph,$$

by multiplicative axiom.

COR. 4. If  $s_1$  or  $s_2$  or ...  $s_r \equiv s$ , where  $s_1, s_2, \dots s_r$  are co-disjunct, and if, further,

$$p/s_1h = p/s_2h = \dots = p/s_rh,$$

then each of these  $= p/sh$ .

For, let each of the above  $= x$ . Then

$$ps/h = ps_1/h + ps_2/h + \dots + ps_r/h,$$

$$\text{i.e. } s/h \cdot p/sh = s_1/h \cdot p/s_1h + s_2/h \cdot p/s_2h + \dots + s_r/h \cdot p/s_rh$$

$$= (s_1/h + s_2/h + \dots + s_r/h) x$$

$$= s/h \cdot x.$$

$$\therefore x = p/sh.$$

Q. E. D.

§ 5. Now the axioms of probability enable us to infer any probability-conclusion *only* from probability-premisses. In other words, the calculus of probability does not enable us to infer any probability-value unless we have some probabilities or probability relations *given*. Such data cannot be supplied by the mathematician. E.g. the rules of arithmetic and the axioms of the probability-calculus are utterly impotent to determine, on the supposed knowledge that the throw of a coin must yield either head or tail and cannot yield both, the probability that it will yield head or that it will yield tail. We must assume that the two co-exclusive and co-exhaustive possibilities are *equally probable*, before we can estimate the probability of either as being a half of certitude. The assumptions ultimately required must be regarded as *Postulates*, and their critical examination will not here be entered upon. The working postulates are such as assert equiprobability amongst alternative possibilities; and constitute what may be called, in Mr Keynes's terminology, *postulates of indifference*.

The precise form of the postulate required in each particular application must be justified by the special nature of the case. We shall immediately lay down the two postulates employed in the theory of education, postponing for the present any further philosophical discussion.

§ 6. The two following postulates in the Theory of Education are concerned with the possible occurrences of the determinates  $p_1, p_2, \dots p_a$  under the determinable  $P$ . The symbols of § 1 are employed.

(1) *Combination-Postulate.*

In a total of  $M$  instances, any proportion, say  $m_1 : m_2 : \dots : m_a$ , where  $m_1 + m_2 + \dots + m_a = M$ , is as likely as any other, prior to any knowledge of the occurrences in question.

(2) *Permutation-Postulate.*

Each of the different orders in which a given proportion  $m_1 : m_2 : \dots : m_a$  for  $M$  instances may be presented is as likely as any other, whatever may have been the previously known orders.

In what follows certitude will be represented by unity.

By (1), The probability of any one proportion in  $M$  instances

$$= \frac{M!}{a(a+1)(a+2) \dots (a+M-1)}.$$

By (2), The probability of any one permutation in which the proportion  $m_1 : m_2 : \dots : m_a$  in  $M$  instances may be presented

$$= \frac{m_1! m_2! m_3! \dots m_a!}{M!}.$$



§ 7. Adopting the notation explained above, these postulates may be symbolised:

$$(1) \quad \mu/h = \frac{M!}{a(a+1) \dots (a+M-1)}.$$

$$(2) \quad \vec{\mu}/\mu h = \frac{m_1! m_2! \dots m_a!}{M!}.$$

Now (see Cor. 3)

$$\vec{\mu}/h = \mu \vec{\mu}/h = \mu/h \times \vec{\mu}/\mu h, \\ \therefore (3) \quad \vec{\mu}/h = \frac{m_1! m_2! \dots m_a!}{a(a+1) \dots (a+M-1)}.$$

Formula (3) gives the prior probability that, in a set of  $M$  instances, the characters  $p_1, p_2, \dots p_a$  under  $P$  shall occur in a determinately assigned sequence in which the proportion and the permutation of these character-manifestations are both fixed.

Taking  $N$  instances (next following the  $M$  instances) presenting the proportion  $n_1 : n_2 : \dots : n_a$ , where  $n_1 + n_2 + \dots + n_a = N$ , the principle of formula (3) may be extended to  $M+N$  instances.

Thus

$$(4) \quad \overrightarrow{\mu + v}/h = \frac{(m_1 + n_1)! (m_2 + n_2)! \dots (m_a + n_a)!}{a(a+1) \dots (a+M+N-1)}.$$

Now

$$\overrightarrow{\mu + v}/h = (\vec{\mu} \text{ and } \vec{v})/h = \vec{\mu}/h \times \vec{v}/\mu h, \text{ by Mult. axiom.}$$

$$\text{Hence (5) } \vec{v}/\mu h = \frac{\overrightarrow{\mu + v}/h}{\vec{\mu}/h} = \frac{(m_1 + n_1)!}{m_1!} \dots \frac{(m_a + n_a)!}{m_a!} \\ \times \frac{1}{(M+a)(M+a+1) \dots (M+N+a-1)}$$

by (3) and (4).

From formula (5) which gives  $\vec{v}/\mu h$  we proceed to find the value of  $v/\mu h$  which will complete our solution.

Let  $\vec{v}_a, \vec{v}_b, \vec{v}_c$ , etc. represent the different possible permutations of the same proportion  $v = n_1 : n_2 : \dots : n_a$ .

The number of these is  $\frac{N!}{n_1! n_2! \dots n_a!}$ .

Now expression (5) is independent of the orders  $\vec{v}_a, \vec{v}_b$ , etc. Hence

$$\vec{v}_a/\mu h = \vec{v}_b/\mu h = \vec{v}_c/\mu h = \text{etc.}$$

Hence, by Cor. 2,

$$(6) \quad v/\mu h = (\vec{v}_a \text{ or } \vec{v}_b \text{ or } \vec{v}_c \text{ or etc.})/\mu h \\ = \frac{N!}{n_1! n_2! \dots n_a!} \cdot \vec{v}_a/\mu h \\ = \frac{N!}{(M+a)(M+a+1) \dots (M+N+a-1)} \\ \times \frac{(m_1 + n_1)!}{m_1! n_1!} \dots \frac{(m_a + n_a)!}{m_a! n_a!}.$$

Again, let  $\vec{\mu}_a, \vec{\mu}_b, \vec{\mu}_c$ , etc. represent the different possible permutations of the proportion

$$\mu = m_1 : m_2 : \dots : m_a.$$

Now expression (6) is independent of the orders  $\vec{\mu}_a, \vec{\mu}_b$ , etc. Hence

$$v/\vec{\mu}_a h = v/\vec{\mu}_b h = v/\vec{\mu}_c h = \text{etc.}$$

Hence, by Cor. 4,

$$(7) \quad v/\mu h = v/(\vec{\mu}_a \text{ or } \vec{\mu}_b \text{ or } \vec{\mu}_c \text{ or etc.}) h \\ = v/\vec{\mu}_a h \\ = \frac{N!}{(M+a)(M+a+1) \dots (M+N+a-1)} \\ \times \frac{(m_1 + n_1)!}{m_1! n_1!} \dots \frac{(m_a + n_a)!}{m_a! n_a!}.$$

This provides the required formula, viz. :—

The probability of any *proposed* proportion in  $N$  unexamined cases as depending upon any *supposed* proportion in  $M$  examined cases.

It will be observed that the highest value of this probability, if  $N=M$ , is given by

$$n_1 = m_1; n_2 = m_2; \dots n_a = m_a,$$

i.e. the most likely proportion for the new cases is the proportion holding of the known cases.

And, generally, the more closely  $\nu$  agrees with  $\mu$ , the greater is the probability that  $\nu$  will be true when  $\mu$  is known to be true.

#### § 8. Elucidation of the formula for $\vec{\nu}/\mu h$ .

As above, we see that  $\vec{\nu}/\mu h = \vec{\nu}/\mu h$ .

Taking  $N$  to be successively 1, 2, 3, etc. the simplicity of the above results will be readily seen.

Thus, for  $N=1$ , the different values of the proposal  $\vec{\nu}$  are  $p_1, p_2, \dots p_a$ . Thus

$$p_1/\mu h = \frac{m_1+1}{M+a}; p_2/\mu h = \frac{m_2+1}{M+a}; \dots p_a/\mu h = \frac{m_a+1}{M+a}.$$

For  $N=2$ , the different values are the dual permutations  $p_1 p_1; p_1 p_2; \dots p_1 p_a; p_2 p_1; p_2 p_2; \dots p_2 p_a$ , etc., etc. Thus

$$\begin{aligned} p_1 p_1/\mu h &= \frac{(m_1+1)(m_1+2)}{(M+a)(M+a+1)}; \\ p_1 p_2/\mu h &= \frac{(m_1+1)(m_2+1)}{(M+a)(M+a+1)}, \text{ etc.,} \\ p_2 p_1/\mu h &= \frac{(m_2+1)(m_1+1)}{(M+a)(M+a+1)}; \\ p_2 p_2/\mu h &= \frac{(m_2+1)(m_2+2)}{(M+a)(M+a+1)}, \text{ etc.,} \\ \text{etc.,} & \qquad \text{etc.,} \qquad \text{etc.} \end{aligned}$$

For  $N=3$ , the probabilities of the triple permutations are:

$$\begin{aligned} p_1 p_1 p_1/\mu h &= \frac{(m_1+1)(m_1+2)(m_1+3)}{(M+a)(M+a+1)(M+a+2)}; \\ p_1 p_1 p_2/\mu h &= \frac{(m_1+1)(m_1+2)(m_2+1)}{(M+a)(M+a+1)(M+a+2)}, \text{ etc.,} \\ & \text{etc.} \end{aligned}$$

By addition of the values for  $N=2$ , we obtain those for  $N=1$ . And, by addition of the values for  $N=3$ , we obtain those for  $N=2$ . And so on. In this way the correctness of each formula is verified.

Moreover, all specific results of the formula giving  $\vec{\nu}/\mu h$  may be schematised—if we typify occurrences as draws from a bag containing an indefinite number of balls of the different colours  $p_1, p_2, \dots p_a$ —by supposing a model bag containing at first  $a$  balls of different colours. As each new ball is drawn from the *real* bag, its colour is observed and it is transferred to the *model* bag. Then, *the probability of any proposed colour being drawn from the real bag is the same as that of its being drawn from the model bag.*

§ 9. The type of case for which the two Postulates are permissible may be thus described.

It is known that there are certain conditions which are *constant* in all the occurrences that may take place and to which our observations and inferences refer. It is also presumed that these permanent or constant conditions are such as *tend* to produce a certain (but unknown) proportion amongst the manifested characters within any given set of  $M$  occurrences. On the other hand, each individual occurrence is actually occasioned by *variable* conditions, which are causally independent

of one another, and are such that no prediction as to their result in any one case can be made.

In such a typical state of affairs, what is unknown is the proportion which tends to be exhibited owing to the unchanged or permanent set of causal conditions. While, therefore, the determinate issue in any set of instances is *causally* independent of what has previously occurred, yet it is *epistemically* dependent; i.e. from the point of view of knowledge, the observation of previously examined instances rationally influences our estimate of probability in regard to what will subsequently occur.

The first postulate, that (in  $M$  cases) one *proportion* is as likely as another, is negatively justified by our ignorance of the proportion which the permanent conditions tend to produce. And our second postulate, that one *order* in which any proportion may be manifested is as likely as any other, is positively justified by our knowledge that the variable conditions which occasion each individual occurrence are ontologically independent of those which occasion any other individual occurrence.

When it is said that the permanent conditions *tend* to produce a certain proportion  $p_1:p_2:\dots:p_a$ , by this it is not meant that such a proportion will be more nearly approached as the series is indefinitely prolonged. For, on the contrary, in  $2M$  cases the proportion  $p_1:p_2:\dots:p_a$  is very much less likely to be exhibited than in  $M$  cases; since the number of arithmetically possible proportions is much greater in a total of  $2M$  than in a total of  $M$ .

We may finally point out that the type of case for which our theory of eduction holds may be figuratively represented by imagining a die, in the form of a solid polyhedron, whose plane faces are not more than  $a$  in number. Moreover, the die is not known to be either

physically or geometrically regular. Each throw of the die represents an occurrence; and, according as the die falls *upon* one or another plane face, we represent the occurrence as being characterised by one or another of the  $a$  possible determinate adjectives— $p_1, p_2, \dots, p_a$ .

The constancy of the physical and geometrical properties of the die corresponds to the *constancy* of those unchanged causal conditions upon which the occurrences depend; while the variable and unassignable impetus of each toss corresponds to the *variable* condition which occasions the actual issue in each individual occurrence:—the varying condition which determines the issue in any one case being causally independent of that which determines the issue in any other case.

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